

# THE STOKES OPERATOR WITH NEUMANN BOUNDARY CONDITIONS IN LIPSCHITZ DOMAINS

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ABSTRACT. In the first part of the paper we give a satisfactory definition of the Stokes operator in Lipschitz domains in  $\mathbb{R}^n$  when boundary conditions of Neumann type are considered. We then proceed to establish optimal global Sobolev regularity results for vector fields in the domains of fractional powers of this Neumann-Stokes operator.

## 1. INTRODUCTION

Let  $\Omega$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and fix a finite number  $T > 0$ . The Navier-Stokes equations are the standard system of PDE's governing the flow of continuum matter in fluid form, such as liquid or gas, occupying the domain  $\Omega$ . These equations describe the change with respect to time  $t \in [0, T]$  of the velocity and pressure of the fluid. A widely used version of the Navier-Stokes initial boundary problem, equipped with a Dirichlet boundary condition, reads

$$(1.1) \quad \left\{ \begin{array}{ll} \frac{\partial \vec{u}}{\partial t} - \Delta_x \vec{u} + \nabla_x \pi + (\vec{u} \cdot \nabla_x) \vec{u} = 0 & \text{in } (0, T] \times \Omega, \\ \operatorname{div}_x \vec{u} = 0 & \text{in } [0, T] \times \Omega, \\ \operatorname{Tr}_x \vec{u} = 0 & \text{on } [0, T] \times \partial\Omega, \\ \vec{u}(0) = \vec{u}_0 & \text{in } \Omega, \end{array} \right.$$

where  $\vec{u}$  is the velocity field and  $\pi$  denotes the pressure of the fluid. One of the strategies for dealing with (1.1), brought to prominence by the pioneering work of H. Fujita, and T. Kato in the 60's, consists of recasting (1.1) in the form of an abstract initial value problem

$$(1.2) \quad \left\{ \begin{array}{l} \vec{u}'(t) + (A\vec{u})(t) = \vec{f}(t) \quad t \in (0, T), \\ \vec{f}(t) := -\mathbb{P}_D [(\vec{u}(t) \cdot \nabla_x) \vec{u}(t)], \\ \vec{u}(0) = \vec{u}_0, \end{array} \right.$$

which is then converted into the integral equation

$$(1.3) \quad \vec{u}(t) = e^{-tA} \vec{u}_0 - \int_0^t e^{-(t-s)A} \mathbb{P}_D [(\vec{u}(s) \cdot \nabla_x) \vec{u}(s)] ds, \quad 0 < t < T,$$

then finally solving (1.3) via fixed point methods (typically, a Picard iterative scheme). In this scenario, the operator  $\mathbb{P}_D$  is the Leray (orthogonal) projection of  $L^2(\Omega)^n$  onto the

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Supported in part by NSF and a UMC Miller Scholar Award  
 2000 *Mathematics Subject Classification*. Primary: 35Q30, 76D07, Secondary: 35A15, 35Q10, 76D03  
*Key words and phrases*. Stokes operator, Neumann boundary conditions, Lipschitz domains, domain of fractional power, regularity, Sobolev spaces, Navier-Stokes equations.

space  $\mathcal{H}_D := \{\vec{u} \in L^2(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega, \nu \cdot \vec{u} = 0 \text{ on } \partial\Omega\}$ , where  $\nu$  is the outward unit normal to  $\Omega$ , and  $A$  is the Stokes operator, i.e. the Friedrichs extension of the symmetric operator  $\mathbb{P}_D \circ (-\Delta_D)$ , where  $\Delta_D$  is the Dirichlet Laplacian, to an unbounded self-adjoint operator on the space  $\mathcal{H}_D$ .

By relying on the theory of analytic semigroups generated by self-adjoint operators, Fujita and Kato have proved in [10] short time existence of strong solutions for (1.1) when  $\Omega \subset \mathbb{R}^3$  is bounded and sufficiently smooth. Somewhat more specifically, they have shown that if  $\Omega$  is a bounded domain in  $\mathbb{R}^3$  with boundary  $\partial\Omega$  of class  $C^3$ , and if the initial datum  $\vec{u}_0$  belongs to  $D(A^{\frac{1}{4}})$ , then a strong solution can be found for which  $\vec{u}(t) \in D(A^{\frac{3}{4}})$  for  $t \in (0, T)$ , granted that  $T$  is small. Hereafter,  $D(A^\alpha)$ ,  $\alpha > 0$ , stands for the domain of the fractional power  $A^\alpha$  of  $A$ .

An important aspect of this analysis is the ability to describe the size/smoothness of vector fields belonging to  $D(A^\alpha)$  in terms of more familiar spaces. For example, the estimates (1.18) and (2.17) in [10] amount to

$$(1.4) \quad D(A^\gamma) \subset C^\alpha(\overline{\Omega})^3 \quad \text{if } \frac{3}{4} < \gamma < 1 \quad \text{and} \quad 0 < \alpha < 2(\gamma - \frac{3}{4}),$$

which plays a key role in [10]. Although Fujita and Kato have proved (1.4) via *ad hoc* methods, it was later realized that a more resourceful and elegant approach to such regularity results is to view them as corollaries of optimal embeddings for  $D(A^\alpha)$ ,  $\alpha > 0$ , into the scale of vector-valued Sobolev (potential) spaces of fractional order,  $L_s^p(\Omega)^3$ ,  $1 < p < \infty$ ,  $s \in \mathbb{R}$ . This latter issue turned out to be intimately linked to the smoothness assumptions made on the boundary of the domain  $\Omega$ . For example, Fujita and Morimoto have proved in [11] that

$$(1.5) \quad \partial\Omega \in C^\infty \implies D(A^\alpha) \subset L_{2\alpha}^2(\Omega)^3, \quad 0 \leq \alpha \leq 1,$$

whereas the presence of a single conical singularity on  $\partial\Omega$  may result in the failure of  $D(A)$  to be included in  $L_2^2(\Omega)^3$ .

The issue of extending the Fujita-Kato approach to the class of Lipschitz domains has been recently resolved in [21]. In the process, several useful global Sobolev regularity results for the vector fields in the fractional powers of the Stokes operator have been established. For example, it has been proved in [21] that for any Lipschitz domain  $\Omega$  in  $\mathbb{R}^3$ ,

$$(1.6) \quad D(A^{\frac{3}{4}}) \subset L_{\frac{3}{p}}^p(\Omega)^3 \quad \forall p > 2,$$

$$(1.7) \quad \forall \alpha > \frac{3}{4} \quad \exists p > 3 \quad \text{such that} \quad D(A^\alpha) \subset L_1^p(\Omega)^3,$$

$$(1.8) \quad D(A^\gamma) = L_{2\gamma, z}^2(\Omega)^3 \cap \mathcal{H}_D, \quad 0 < \gamma < \frac{3}{4},$$

where, if  $s > 0$ ,  $L_{s, z}^2(\Omega)$  is the subspace of  $L_s^2(\Omega)$  consisting of functions whose extension by zero outside  $\Omega$  belongs to  $L_s^2(\mathbb{R}^n)$ . Also, it was shown in [21] that for any Lipschitz domain  $\Omega$  in  $\mathbb{R}^3$  there exists  $\varepsilon = \varepsilon(\Omega) > 0$  such that

$$(1.9) \quad \frac{3}{4} < \gamma < \frac{3}{4} + \varepsilon \implies D(A^\gamma) \subset C^{2\gamma-3/2}(\overline{\Omega})^3,$$

in agreement with the Fujita-Kato regularity result (1.4).

The aim of this paper is to derive analogous results in the case when Neumann-type boundary conditions are considered in place of the Dirichlet boundary condition. Dictated by specific practical considerations, several scenarios are possible. For example, the ‘no-slip’ Neumann condition

$$(1.10) \quad (\nabla \vec{u} + \nabla \vec{u}^\top) \nu - \pi \nu = 0 \quad \text{on } (0, T) \times \partial\Omega,$$

(recall that  $\nu$  stands for the outward unit normal to  $\partial\Omega$ ) has been frequently used in the literature. See, e.g., [28], [12] and the references therein. Another Neumann-type condition of interest is

$$(1.11) \quad (\nabla \vec{u}) \nu - \pi \nu = 0 \quad \text{on } (0, T) \times \partial\Omega.$$

This has been employed in [7] (in the stationary case). Here we shall work with a one-parameter family of Neumann-type boundary conditions,

$$(1.12) \quad [(\nabla \vec{u}) + \lambda (\nabla \vec{u})^\top] \nu - \pi \nu = 0 \quad \text{on } (0, T) \times \partial\Omega,$$

indexed by  $\lambda \in (-1, 1]$  (in this context, (1.10), (1.11) correspond to choosing  $\lambda = 1$  and  $\lambda = 0$ , respectively). Much as in the case of the Fujita-Kato approach for (1.1), a basic ingredient in the treatment of the initial Navier-Stokes boundary problem with Neumann boundary conditions, i.e.,

$$(1.13) \quad \begin{cases} \frac{\partial \vec{u}}{\partial t} - \Delta_x \vec{u} + \nabla_x \pi + (\vec{u} \cdot \nabla_x) \vec{u} = 0 & \text{in } (0, T] \times \Omega, \\ \operatorname{div}_x \vec{u} = 0 & \text{in } [0, T] \times \Omega, \\ [(\nabla_x \vec{u}) + \lambda (\nabla_x \vec{u})^\top] \nu - \pi \nu = 0 & \text{on } [0, T] \times \partial\Omega, \\ \vec{u}(0) = \vec{u}_0 & \text{in } \Omega, \end{cases}$$

is a suitable analogue of the Stokes operator  $A = \mathbb{P}_D \circ (-\Delta_D)$  discussed earlier. As a definition for this, we propose taking the unbounded operator

$$(1.14) \quad B_\lambda : D(B_\lambda) \subset \mathcal{H}_N \longrightarrow \mathcal{H}_N,$$

where we have set  $\mathcal{H}_N := \{\vec{u} \in L^2(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega\}$ , with domain

$$(1.15) \quad \begin{aligned} D(B_\lambda) &:= \left\{ \vec{u} \in L^2_1(\Omega)^n \cap \mathcal{H}_N : \text{there exists } \pi \in L^2(\Omega) \text{ so that } -\Delta \vec{u} + \nabla \pi \in \mathcal{H}_N \right. \\ &\quad \left. \text{and such that } [(\nabla \vec{u}) + \lambda (\nabla \vec{u})^\top] \nu - \pi \nu = 0 \text{ on } \partial\Omega \right\}, \end{aligned}$$

(with a suitable interpretation of the boundary condition) and acting according to

$$(1.16) \quad B_\lambda \vec{u} := -\Delta \vec{u} + \nabla \pi, \quad \vec{u} \in D(B_\lambda),$$

In order to be able to differentiate this from the much more commonly used Stokes operator  $A = \mathbb{P}_D \circ (-\Delta_D)$ , we shall call the latter the *Dirichlet-Stokes operator* and refer to (1.15)-(1.16) as the *Neumann-Stokes operator*.

Let us now comment on the suitability of the Neumann-Stokes operator  $B_\lambda$  vis-a-vis to the solvability of the initial Navier-Stokes system with Neumann boundary conditions (1.13). To this end, denote by  $\mathbb{P}_N$  the orthogonal projection of  $L^2(\Omega)^n$  onto the space  $\mathcal{H}_N = \{\vec{u} \in L^2(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega\}$ . In particular,

$$(1.17) \quad \mathbb{P}_N(\nabla q) = 0 \quad \text{for every } q \in L^2_1(\Omega) \text{ with } \operatorname{Tr} q = 0 \text{ on } \partial\Omega.$$

Proceed formally and assume that  $\vec{u}$ ,  $\pi$  solve (1.13) and that  $q$  solves the inhomogeneous Dirichlet problem

$$(1.18) \quad \begin{cases} \Delta q = \Delta \pi \text{ in } \Omega, \\ q|_{\partial\Omega} = 0. \end{cases}$$

Then  $\nabla\pi - \nabla q$  is divergence-free. Based on this and (1.17) we may then compute

$$(1.19) \quad \mathbb{P}_N(\nabla\pi) = \mathbb{P}_N(\nabla\pi - \nabla q) = \nabla\pi - \nabla q = \nabla(\pi - q).$$

Since  $\pi - q$  has the same boundary trace as  $\pi$ , it follows that  $[(\nabla\vec{u}) + \lambda(\nabla\vec{u})^\top]\nu - (\pi - q)\nu = 0$  on  $\partial\Omega$ . Consequently,

$$(1.20) \quad B_\lambda(\vec{u}) = -\Delta \vec{u} + \nabla(\pi - q) = \mathbb{P}_N(-\Delta \vec{u} + \nabla\pi).$$

Thus, when  $\mathbb{P}_N$  is formally applied to the first line in (1.13) we arrive at the abstract evolution problem

$$(1.21) \quad \begin{cases} \vec{u}'(t) + (B_\lambda\vec{u})(t) = \vec{f}(t) & t \in (0, T), \\ \vec{f}(t) := -\mathbb{P}_N[(\vec{u}(t) \cdot \nabla_x)\vec{u}(t)], \\ \vec{u}(0) = \vec{u}_0, \end{cases}$$

which is the natural analogue of (1.2) in the current setting. This opens the door for solving (1.13) by considering the integral equation

$$(1.22) \quad \vec{u}(t) = e^{-tB_\lambda}\vec{u}_0 - \int_0^t e^{-(t-s)B_\lambda}\mathbb{P}_N[(\vec{u}(s) \cdot \nabla_x)\vec{u}(s)] ds, \quad 0 < t < T.$$

In summary, the interest in the functional analytic properties of the Neumann-Stokes operator  $B_\lambda$  in (1.15)-(1.16) is justified. In order to prevent the current paper from becoming too long, we choose to treat the solvability of (1.22) in a separate publication (cf. [22]) and confine ourselves here to establishing sharp global Sobolev regularity results for vector fields in  $D(B_\lambda^\alpha)$ , the domain of fractional powers of  $B_\lambda$ .

Our main results in this regard parallel those for the Dirichlet-Stokes operator which have been reviewed in the first part of the introduction. For the sake of this introduction, we wish to single out several such results. Concretely, for a Lipschitz domain  $\Omega$  in  $\mathbb{R}^n$  we show that

$$(1.23) \quad D(B_\lambda^{\frac{s}{2}}) = \left\{ \vec{u} \in L^2_s(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega \right\} \text{ if } 0 \leq s \leq 1,$$

and

$$(1.24) \quad D(B_\lambda^\alpha) \subset \bigcup_{p > \frac{2n}{n-1}} L_1^p(\Omega)^n \quad \text{if } \alpha > \frac{3}{4}.$$

Also, when  $n = 3$ ,

$$(1.25) \quad D(B_\lambda^\alpha) \subset C^{2\alpha-3/2}(\bar{\Omega})^3 \quad \text{if } \frac{3}{4} < \alpha < \frac{3}{4} + \varepsilon,$$

$$(1.26) \quad D(B_\lambda^{\frac{3}{4}}) \subset L_1^3(\Omega)^3,$$

and when  $n = 2$ ,

$$(1.27) \quad D(B_\lambda^\alpha) \subset C^{2\alpha-1}(\bar{\Omega})^2 \quad \text{if } \frac{3}{4} < \alpha < \frac{3}{4} + \varepsilon,$$

for some small  $\varepsilon = \varepsilon(\Omega) > 0$ .

It should be noted that, in the case when  $\partial\Omega \in C^\infty$ , the initial boundary value problem (1.13) has been treated (when  $\lambda = 1$ ) by G. Grubb in [12]. In this scenario, the departure point is the regularity result  $D(B_1) \subset L_2^2(\Omega)^n$ , which nonetheless is utterly false in the class of Lipschitz domains considered here.

Key ingredients in the proof of the regularity results (1.23)-(1.26) are the sharp results for the well-posedness of the inhomogeneous problem for the Stokes operator equipped with Neumann boundary conditions in a Lipschitz domain  $\Omega$  in  $\mathbb{R}^n$ , with data from Besov and Triebel-Lizorkin spaces from [23]. This yields a clear picture of the nature of  $D(B_\lambda)$ . On the other hand, known abstract functional analytic results allow us to identify  $D(B_\lambda^{1/2})$ . Starting from these, other intermediate fractional powers can then be treated by relying on certain (non-standard) interpolation techniques.

The organization of the paper is as follows. In Section 2 we collect a number of preliminary results of function theoretic nature. Section 3 is devoted to a discussion of the meaning and properties of the conormal derivative  $[(\nabla \vec{u}) + \lambda (\nabla \vec{u})^\top] \nu - \pi \nu$  on  $\partial\Omega$  when  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain and  $\vec{u}$ ,  $\pi$  belong to certain Besov-Triebel-Lizorkin spaces. Section 4 is reserved for a review of the definitions and properties of linear operators associated with sesquilinear forms. Next, in Section 5, we collect some basic abstract results about semigroups and fractional powers of self-adjoint operators. The rigorous definition of the Neumann-Stokes operator  $B_\lambda$  is given in Section 6. Among other things, here we show that  $B_\lambda$  is self-adjoint on  $\mathcal{H}_N$  and identify  $D(B_\lambda^{1/2})$ . The scale  $V^{p,s}(\Omega) := \{\vec{u} \in L_s^p(\Omega)^n : \operatorname{div} \vec{u} = 0\}$  is investigated in Section 7 where we show that, for certain ranges of indices, this is stable under complex interpolation and duality. In Section 8 we record an optimal, well-posedness result for the Poisson problem for the Stokes system with Neumann-type boundary conditions in Lipschitz domains, with data from Besov-Triebel-Lizorkin spaces, recently established in [23]. Finally, in Section 9 and Section 10, we investigate the global Sobolev regularity of vector fields belonging to  $D(B_\lambda^\alpha)$  for  $\alpha \in [0, 1]$ , when the underlying domain is Lipschitz.

*Acknowledgments.* This work has been completed while the authors had been visiting Université Aix-Marseille 3 and the University of Missouri-Columbia, whose hospitality they wish to gratefully acknowledge. The first named author is also greatly indebted to

Fritz Gesztesy for many helpful discussions pertaining to the abstract operator theory used in this paper.

## 2. PRELIMINARIES

We shall call an open, bounded, nonempty set, with connected boundary  $\Omega \subset \mathbb{R}^n$  a *Lipschitz domain* if for every point  $x^* \in \partial\Omega$  there is a rotation of the Euclidean coordinates in  $\mathbb{R}^n$ , a neighborhood  $\mathcal{O}$  of  $x^*$  and a Lipschitz function  $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  such that

$$(2.1) \quad \Omega \cap \mathcal{O} = \{x = (x', x_n) \in \mathbb{R}^n : x_n > \varphi(x')\} \cap \mathcal{O}.$$

In this scenario, we let  $d\sigma$  stand for the surface measure on  $\partial\Omega$ , and denote by  $\nu$  the outward unit normal to  $\partial\Omega$ . Next, for  $k \in \mathbb{N}$  and  $p \in (1, \infty)$ , we recall the classical Sobolev space

$$(2.2) \quad L_k^p(\Omega) := \left\{ f \in L^p(\Omega) : \|f\|_{W^{k,p}(\Omega)} := \sum_{|\gamma| \leq k} \|\partial^\gamma f\|_{L^p(\Omega)} < \infty \right\},$$

and set

$$(2.3) \quad L_{k,z}^p(\Omega) := \text{the closure of } C_c^\infty(\Omega) \text{ in } L_k^p(\Omega).$$

Then for every  $k \in \mathbb{N}$  and  $1 < p, p' < \infty$  with  $1/p + 1/p' = 1$ , we have

$$(2.4) \quad L_{-k}^p(\Omega) := \left\{ \sum_{|\gamma| \leq k} \partial^\gamma f_\gamma : f_\gamma \in L^p(\Omega) \right\} = \left( L_{k,z}^{p'}(\Omega) \right)^*.$$

Moving on, for  $s \in (0, 1)$ ,  $1 \leq p \leq \infty$ , denote by

$$(2.5) \quad B_s^{p,p}(\partial\Omega) := \left\{ f \in L^p(\partial\Omega) : \left( \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n-1+sp}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}} < +\infty \right\},$$

the Besov class on  $\partial\Omega$ . We equip this with the natural norm  $\|f\|_{B_s^{p,p}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \left( \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n-1+sp}} d\sigma_x d\sigma_y \right)^{1/p}$ . For  $s \in (0, 1)$  and  $1 < p, p' < \infty$  with  $1/p + 1/p' = 1$ , we also set

$$(2.6) \quad B_{-s}^{p,p}(\partial\Omega) := \left( B_s^{p',p'}(\partial\Omega) \right)^*.$$

In the sequel, we shall occasionally write  $L_s^2(\partial\Omega)$  in place of  $B_s^{2,2}(\partial\Omega)$  for  $s \in (-1, 1)$ .

Recall that if  $p \in (1, \infty)$  and  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain, then the trace operator

$$(2.7) \quad \text{Tr} : L_1^p(\Omega) \longrightarrow B_{1-1/p}^{p,p}(\partial\Omega)$$

is well-defined, linear and bounded.

Next, introduce

$$(2.8) \quad \mathcal{H} := \{ \vec{u} \in L^2(\Omega)^n : \text{div } \vec{u} = 0 \text{ in } \Omega \}$$

which is a closed subspace of  $L^2(\Omega)^n$  (hence, a Hilbert space when equipped with the norm inherited from  $L^2(\Omega)^n$ ). Also, set

$$(2.9) \quad \mathcal{V} := L_1^2(\Omega)^n \cap \mathcal{H}$$

which is a closed subspace of  $L_1^2(\Omega)^n$  hence, a reflexive Banach space when equipped with the norm inherited from  $L_1^2(\Omega)^n$ .

**Lemma 2.1.** *If  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain then*

$$(2.10) \quad \mathcal{V} \hookrightarrow \mathcal{H} \text{ continuously and densely.}$$

*Proof.* The continuity of the inclusion mapping in (2.10) is obvious. To prove that this has a dense range, fix  $\vec{u} \in \mathcal{H}$ . Then it has been proved in [16] that there exists a smooth domain  $\mathcal{O}$  and  $\vec{w} \in L^2(\mathcal{O})^n$  with the following properties:

$$(2.11) \quad \bar{\Omega} \subset \mathcal{O}, \quad \operatorname{div} \vec{w} = 0 \text{ in } \mathcal{O}, \quad \vec{w}|_{\Omega} = \vec{u}.$$

In analogy with (2.8), (2.9), set

$$(2.12) \quad \mathcal{H}(\mathcal{O}) := \{\vec{v} \in L^2(\mathcal{O})^n : \operatorname{div} \vec{v} = 0 \text{ in } \mathcal{O}\}, \quad \mathcal{V}(\mathcal{O}) := L_1^2(\mathcal{O})^n \cap \mathcal{H}(\mathcal{O}).$$

Then the following Hodge-Helmholtz-Weyl decompositions are valid

$$(2.13) \quad L_1^2(\mathcal{O})^n = \mathcal{V}(\mathcal{O}) \oplus \nabla \left[ L_2^2(\mathcal{O}) \cap L_{1,z}^2(\mathcal{O}) \right],$$

$$(2.14) \quad L^2(\mathcal{O})^n = \mathcal{H}(\mathcal{O}) \oplus \left[ \nabla L_{1,z}^2(\mathcal{O}) \right].$$

These can be obtained constructively as follows. Granted that  $\mathcal{O}$  is a smooth domain (here, it suffices to have  $\partial\mathcal{O} \in C^{1,r}$  for some  $r > 1/2$ ), the Poisson problem with homogeneous Dirichlet boundary condition

$$(2.15) \quad \begin{cases} \Delta q = f \in L^2(\mathcal{O}), \\ q \in L_2^2(\mathcal{O}) \cap L_{1,z}^2(\mathcal{O}), \end{cases}$$

is well-posed, and we denote by

$$(2.16) \quad G : L^2(\mathcal{O}) \longrightarrow L_2^2(\mathcal{O}) \cap L_{1,z}^2(\mathcal{O}), \quad Gf = q,$$

the solution operator associated with (2.15). By the Lax-Milgram lemma, the latter further extends to a bounded, self-adjoint operator

$$(2.17) \quad G : L_{-1}^2(\mathcal{O}) \longrightarrow L_{1,z}^2(\mathcal{O}).$$

With  $I$  denoting the identity operator, if we now consider

$$(2.18) \quad P := I - \nabla \circ G \circ \operatorname{div},$$

then in each instance below

$$(2.19) \quad P : L_1^2(\mathcal{O})^n \longrightarrow \mathcal{V}(\mathcal{O}), \quad P : L^2(\mathcal{O})^n \longrightarrow \mathcal{H}(\mathcal{O}),$$

$P$  is a well-defined, linear and bounded operator. Furthermore, in the second case in (2.19),  $P$  actually acts as the orthogonal projection. Indeed, this is readily verified using the fact that

$$(2.20) \quad P = P^* \text{ in } L^2(\mathcal{O})^n \quad \text{and} \quad P|_{\mathcal{H}(\mathcal{O})} = I, \text{ the identity operator.}$$

The Hodge-Helmholtz-Weyl decompositions (2.13)-(2.14) are then naturally induced by decomposing the identity operator according to

$$(2.21) \quad I = P + \nabla \circ G \circ \text{div},$$

both on  $L_1^2(\mathcal{O})^n$  and on  $L^2(\mathcal{O})^n$ .

After this preamble, we now turn to the task of establishing (2.10). Choose a sequence  $\vec{w}_j \in L_1^2(\mathcal{O})^n$ ,  $j \in \mathbb{N}$ , such that  $\vec{w}_j \rightarrow \vec{w}$  in  $L^2(\mathcal{O})^n$  as  $j \rightarrow \infty$ . Then  $\vec{w} = P\vec{w} = \lim_{j \rightarrow \infty} P\vec{w}_j$  in  $L^2(\mathcal{O})^n$  and  $\vec{u}_j := [P\vec{w}_j]|_{\Omega} \in \mathcal{V}$  for every  $j \in \mathbb{N}$ . Since these considerations imply that  $\vec{u} = \vec{w}|_{\Omega} = \lim_{j \rightarrow \infty} \vec{u}_j$  in  $L^2(\Omega)^n$ , (2.10) follows.  $\square$

**Remark.** An inspection of the above proof shows that, via a similar argument, we have that

$$(2.22) \quad P : C^\infty(\bar{\Omega}) \hookrightarrow \mathcal{H} \cap C^\infty(\bar{\Omega}) \quad \text{boundedly.}$$

Thus, ultimately,

$$(2.23) \quad \{\vec{u} \in C^\infty(\bar{\Omega})^n : \text{div } \vec{u} = 0 \text{ in } \Omega\} \hookrightarrow \mathcal{H} \quad \text{densely.}$$

Next, we introduce the following closed subspace of  $L_{1/2}^2(\partial\Omega)^n$ :

$$(2.24) \quad L_{1/2,\nu}^2(\partial\Omega) := \left\{ \vec{\varphi} \in L_{1/2}^2(\partial\Omega)^n : \int_{\partial\Omega} \nu \cdot \vec{\varphi} \, d\sigma = 0 \right\}.$$

Our goal is to show that the trace operator from (2.7) extends to a bounded mapping

$$(2.25) \quad \text{Tr} : \mathcal{V} \longrightarrow L_{1/2,\nu}^2(\partial\Omega)$$

which is onto. In fact, it is useful to prove the following more general result.

**Lemma 2.2.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain, with outward unit normal  $\nu$  and surface measure  $d\sigma$ . Also, fix  $1 < p < \infty$  and  $s \in (1/p, 1 + 1/p)$ . Then the trace operator from (2.7) extends to a bounded mapping*

$$(2.26) \quad \text{Tr} : \left\{ \vec{u} \in L_s^p(\Omega)^n : \text{div } \vec{u} = 0 \right\} \longrightarrow \left\{ \vec{\varphi} \in B_{s-1/p}^{p,p}(\partial\Omega)^n : \int_{\partial\Omega} \nu \cdot \vec{\varphi} \, d\sigma = 0 \right\},$$

which is onto.



*Proof.* The fact that (2.26) is well-defined, linear and bounded is clear from the properties of (2.7) and the fact that

$$(2.27) \quad \int_{\partial\Omega} \nu \cdot \text{Tr } \vec{u} \, d\sigma = \int_{\Omega} \text{div } \vec{u} \, dx = 0,$$

whenever  $\vec{u} \in L_s^p(\Omega)^n$  is divergence-free. To see that (2.26) is also onto, consider  $\vec{\varphi} \in B_{s-1/p}^{p,p}(\partial\Omega)^n$  satisfying

$$(2.28) \quad \int_{\partial\Omega} \nu \cdot \vec{\varphi} \, d\sigma = 0$$

and solve the divergence equation

$$(2.29) \quad \begin{cases} \text{div } \vec{u} = 0 \text{ in } \Omega, \\ \vec{u} \in L_s^p(\Omega)^n, \\ \text{Tr } \vec{u} = \vec{\varphi} \text{ on } \partial\Omega. \end{cases}$$

For a proof of the fact that this is solvable for any  $\vec{\varphi} \in B_{s-1/p}^{p,p}(\partial\Omega)^n$  satisfying (2.28) see [19]. This shows that the operator (2.26) is indeed onto.  $\square$

Moving on, for  $\lambda \in \mathbb{R}$  fixed, let

$$(2.30) \quad a_{jk}^{\alpha\beta}(\lambda) := \delta_{jk}\delta_{\alpha\beta} + \lambda \delta_{j\beta}\delta_{k\alpha}, \quad 1 \leq j, k, \alpha, \beta \leq n,$$

and, adopting the summation convention over repeated indices, consider the differential operator  $L_\lambda$  given by

$$(2.31) \quad (L_\lambda \vec{u})_\alpha := \partial_j (a_{jk}^{\alpha\beta}(\lambda) \partial_k u_\beta) = \Delta u_\alpha + \lambda \partial_\alpha (\text{div } \vec{u}), \quad 1 \leq \alpha \leq n.$$

Next, assuming that  $\lambda \in \mathbb{R}$  and  $\vec{u}, \pi$  are sufficiently nice functions in a Lipschitz domain  $\Omega \subset \mathbb{R}^n$  with outward unit normal  $\nu$ , define the conormal derivative

$$(2.32) \quad \begin{aligned} \partial_\nu^\lambda(\vec{u}, \pi) &:= \left( \nu_j a_{jk}^{\alpha\beta}(\lambda) \partial_k u_\beta - \nu_\alpha \pi \right)_{1 \leq \alpha \leq n} \\ &= \left[ (\nabla \vec{u})^\top + \lambda (\nabla \vec{u}) \right] \nu - \pi \nu \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\nabla \vec{u} = (\partial_k u_j)_{1 \leq j, k \leq n}$  denotes the Jacobian matrix of the vector-valued function  $\vec{u}$ , and  $\top$  stands for transposition of matrices. Introducing the bilinear form

$$(2.33) \quad A_\lambda(\xi, \zeta) := a_{jk}^{\alpha\beta}(\lambda) \xi_j^\alpha \zeta_k^\beta, \quad \forall \xi, \zeta \text{ } n \times n \text{ matrices},$$

we then have the following useful integration by parts formula:

$$(2.34) \quad \int_{\Omega} \langle L_\lambda \vec{u} - \nabla \pi, \vec{w} \rangle \, dx = \int_{\partial\Omega} \langle \partial_\nu^\lambda(\vec{u}, \pi), \vec{w} \rangle \, d\sigma - \int_{\Omega} \left\{ A_\lambda(\nabla \vec{u}, \nabla \vec{w}) - \pi(\text{div } \vec{w}) \right\} \, dx.$$

In turn, this readily implies that

$$\begin{aligned}
\int_{\Omega} \langle L_{\lambda} \vec{u} - \nabla \pi, \vec{w} \rangle dx - \int_{\Omega} \langle L_{\lambda} \vec{w} - \nabla \rho, \vec{u} \rangle dx &= \int_{\partial\Omega} \left\{ \langle \partial_{\nu}^{\lambda}(\vec{u}, \pi), \vec{w} \rangle - \langle \partial_{\nu}^{\lambda}(\vec{w}, \rho), \vec{u} \rangle \right\} d\sigma \\
(2.35) \qquad \qquad \qquad &+ \int_{\Omega} \left\{ \pi(\operatorname{div} \vec{w}) - \rho(\operatorname{div} \vec{u}) \right\} dx.
\end{aligned}$$

Above, it is implicitly assumed that the functions involved are reasonably behaved near the boundary. Such considerations are going to be paid appropriate attention to in each specific application of these integration by parts formulas.

### 3. CONORMAL DERIVATIVE IN BESOV-TRIEBEL-LIZORKIN SPACES

For  $0 < p, q \leq \infty$  and  $s \in \mathbb{R}$ , we denote the Besov and Triebel-Lizorkin scales in  $\mathbb{R}^n$  by  $B_s^{p,q}(\mathbb{R}^n)$  and  $F_s^{p,q}(\mathbb{R}^n)$ , respectively (cf., e.g., [30]). Next, given  $\Omega \subset \mathbb{R}^n$  Lipschitz domain and  $0 < p, q \leq \infty$ ,  $\alpha \in \mathbb{R}$ , we set

$$\begin{aligned}
A_{\alpha}^{p,q}(\Omega) &:= \{u \in \mathcal{D}'(\Omega) : \exists v \in A_{\alpha}^{p,q}(\mathbb{R}^n) \text{ with } v|_{\Omega} = u\}, \\
(3.1) \quad A_{\alpha,0}^{p,q}(\Omega) &:= \{u \in A_{\alpha}^{p,q}(\mathbb{R}^n) \text{ with } \operatorname{supp} u \subseteq \overline{\Omega}\}, \\
A_{\alpha,z}^{p,q}(\Omega) &:= \{u|_{\Omega} : u \in A_{\alpha,0}^{p,q}(\Omega)\},
\end{aligned}$$

where  $A \in \{B, F\}$ . Finally, we let  $B_s^{p,q}(\partial\Omega)$  stand for the Besov class on the Lipschitz manifold  $\partial\Omega$ , obtained by transporting (via a partition of unity and pull-back) the standard scale  $B_s^{p,q}(\mathbb{R}^{n-1})$ . We shall frequently use the abbreviation

$$(3.2) \quad L_s^p(\Omega) := F_s^{p,2}(\Omega), \quad 1 < p < \infty, \quad s \in \mathbb{R}.$$

As is well-known, this is consistent with (2.2) and (2.4).

The existence of a universal linear extension operator, from Lipschitz domains to the entire Euclidean space, which preserves smoothness both on the Besov and the Triebel-Lizorkin scales has been established by V. Rychkov. In [27], he proved the following:

**Theorem 3.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a Lipschitz domain. Then there exists a linear, continuous operator  $E_{\Omega}$ , mapping distributions in  $\Omega$  into tempered distributions in  $\mathbb{R}^n$ , such that whenever  $0 < p, q \leq +\infty$ ,  $s \in \mathbb{R}^n$ , then*

$$(3.3) \quad E_{\Omega} : A_s^{p,q}(\Omega) \longrightarrow A_s^{p,q}(\mathbb{R}^n) \text{ boundedly, satisfying } \mathcal{R}_{\Omega} \circ E_{\Omega} f = f, \quad \forall f \in A_s^{p,q}(\Omega),$$

for  $A = B$  or  $A = F$ , in the latter case assuming  $p < \infty$ .

Let us also record here a useful lifting result for fractional order Sobolev spaces on Lipschitz domains, which has been proved in [19].

**Proposition 3.2.** *Let  $1 < p < \infty$  and  $s \in \mathbb{R}$ . Then for any distribution  $u$  in the Lipschitz domain  $\Omega \subset \mathbb{R}^n$ , the following implication holds:*

$$(3.4) \quad \nabla u \in L_{s-1}^p(\Omega, \mathbb{R}^n) \implies u \in L_s^p(\Omega).$$

The following useful consequence of Proposition 3.2 (cf. [23] for a direct proof) will be used frequently in this paper.

**Corollary 3.3.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a Lipschitz domain and suppose that  $1 < p < \infty$ . Then there exists a finite constant  $C > 0$  depending only on  $n$ ,  $p$ , and the Lipschitz character of  $\Omega$  such that every distribution  $u \in L^p_{-1}(\Omega)$  with  $\nabla u \in L^p_{-1}(\Omega)^n$  has the property that  $u \in L^p(\Omega)$  and*

$$(3.5) \quad \|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p_{-1}(\Omega)^n} + C \operatorname{diam}(\Omega) \|u\|_{L^p_{-1}(\Omega)}$$

holds.

Later on, we shall need duality results for the scales introduced at the beginning of this section. Throughout, all duality pairings on  $\Omega$  are extensions of the natural pairing between test functions and distributions on  $\Omega$ . As far as the nature of the dual of  $L^p_s(\Omega)$  is concerned, when  $1 < p, p' < \infty$ ,  $1/p + 1/p' = 1$  and  $s \in \mathbb{R}$  we have that

$$(3.6) \quad \begin{aligned} & \widetilde{C_c^\infty(\Omega)} \ni \tilde{\varphi} \longmapsto \varphi \in C_c^\infty(\Omega) \text{ extends to} \\ & \text{an isomorphism } \Psi : L^p_{s,0}(\Omega) \longrightarrow \left(L^{p'}_{-s}(\Omega)\right)^*. \end{aligned}$$

Other duality results of interest are

$$(3.7) \quad \left(L^p_{s,z}(\Omega)\right)^* = L^{p'}_{-s}(\Omega) \quad \text{if } 1 < p < \infty \text{ and } s > -1 + \frac{1}{p},$$

and

$$(3.8) \quad \left(L^p_s(\Omega)\right)^* = L^{p'}_{-s,z}(\Omega) \quad \text{if } 1 < p < \infty \text{ and } s < \frac{1}{p}.$$

In particular,

$$(3.9) \quad \left(L^p_s(\Omega)\right)^* = L^{p'}_{-s}(\Omega), \quad \forall s \in (-1 + 1/p, 1/p).$$

See the discussion in [21].

Moving on, we shall need a refinement of (2.7) in the context of Besov and Triebel-Lizorkin spaces. To state this result, let  $(a)_+ := \max\{a, 0\}$ .

**Proposition 3.4.** *Let  $\Omega$  be a Lipschitz domain in  $\mathbb{R}^n$  and assume that the indices  $p, s$  satisfy  $\frac{n-1}{n} < p \leq \infty$  and  $(n-1)(\frac{1}{p} - 1)_+ < s < 1$ . Then the following hold:*

(i) *The restriction to the boundary extends to a linear, bounded operator*

$$(3.10) \quad \operatorname{Tr} : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_s(\partial\Omega) \quad \text{for } 0 < q \leq \infty.$$

For this range of indices,  $\operatorname{Tr}$  is onto and has a bounded right inverse

$$(3.11) \quad \operatorname{Ex} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega).$$

As far as the null-space of (3.10) is concerned, if  $\frac{n-1}{n} < p < \infty$ ,  $(n-1)(1/p - 1)_+ < s < 1$  and  $0 < q < \infty$ , then

$$(3.12) \quad B_{s+1/p,z}^{p,q}(\Omega) = \left\{ u \in B_{s+1/p}^{p,q}(\Omega) : \text{Tr } u = 0 \right\},$$

and

$$(3.13) \quad C_c^\infty(\Omega) \hookrightarrow B_{s+1/p,z}^{p,q}(\Omega) \text{ densely.}$$

(ii) Similar considerations hold for

$$(3.14) \quad \text{Tr} : F_{s+\frac{1}{p}}^{p,q}(\Omega) \longrightarrow B_s^{p,p}(\partial\Omega)$$

with the convention that  $q = \infty$  if  $p = \infty$ . More specifically,  $\text{Tr}$  in (3.14) is a linear, bounded, operator which has a linear, bounded right inverse

$$(3.15) \quad \text{Ex} : B_s^{p,p}(\partial\Omega) \longrightarrow F_{s+\frac{1}{p}}^{p,q}(\Omega).$$

Also, if  $\frac{n-1}{n} < p < \infty$ ,  $(n-1)(1/p-1)_+ < s < 1$  and  $\min\{1, p\} \leq q < \infty$ , then

$$(3.16) \quad F_{s+1/p,z}^{p,q}(\Omega) = \left\{ u \in F_{s+1/p}^{p,q}(\Omega) : \text{Tr } u = 0 \right\},$$

and

$$(3.17) \quad C_c^\infty(\Omega) \hookrightarrow F_{s+1/p,z}^{p,q}(\Omega) \text{ densely.}$$

Let  $X$  be a Banach space with dual  $X^*$ . For every  $n \times n$  matrix  $F = (F_j^\alpha)_{\alpha,j}$  with entries from  $X$ , every  $n \times n$  matrix  $G = (G_k^\beta)_{\beta,k}$  with entries from  $X^*$ , and each  $\lambda \in \mathbb{R}$ , we set

$$(3.18) \quad \mathbb{A}_\lambda^X(F, G) := a_{jk}^{\alpha\beta}(\lambda) {}_X \langle F_j^\alpha, G_k^\beta \rangle_{X^*},$$

where  ${}_X \langle \cdot, \cdot \rangle_{X^*}$  is the duality pairing between  $X$  and  $X^*$ , and  $a_{jk}^{\alpha\beta}(\lambda)$  are as in (2.30). In the sequel, our notation will not emphasize the dependence of  $\langle \cdot, \cdot \rangle$  and  $\mathbb{A}_\lambda$  on  $X$ ; however, the particular nature of  $X$  should be clear from the context in each case.

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded Lipschitz domain and assume that  $1 < p, q < \infty$ ,  $0 < s < 1$ . If  $\vec{u} \in B_{s+\frac{1}{p}}^{p,q}(\Omega)^n$ ,  $\pi \in B_{s+\frac{1}{p}-1}^{p,q}(\Omega)$  and  $\vec{f} \in B_{s+\frac{1}{p}-2,0}^{p,q}(\Omega)^n$  are such that  $\Delta \vec{u} - \nabla \pi = \vec{f}|_\Omega$  in  $\Omega$ , then as suggested by (2.35), it is natural to define

$$(3.19) \quad \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}} \in B_{s-1}^{p,q}(\partial\Omega) = \left( B_{1-s}^{p',q'}(\partial\Omega) \right)^*, \quad 1/p + 1/p' = 1, \quad 1/q + 1/q' = 1, \quad \lambda \in \mathbb{R},$$

by setting

$$(3.20) \quad \begin{aligned} \left\langle \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}}, \vec{\psi} \right\rangle &:= \left\langle \vec{f}, \text{Ex}(\vec{\psi}) \right\rangle + \mathbb{A}_\lambda \left( \nabla \vec{u}, \nabla \text{Ex}(\vec{\psi}) \right) \\ &\quad - \left\langle \pi, \text{div Ex}(\vec{\psi}) \right\rangle, \quad \forall \vec{\psi} \in B_{1-s}^{p',q'}(\partial\Omega)^n, \end{aligned}$$

where  $\text{Ex}$  is the extension operator introduced in Proposition 3.4. The conditions on the indices  $p, q, s$  ensure that all duality pairings in the right-hand side of (3.20) are well-defined. Similar considerations apply to the case when  $\vec{u}, \pi, \vec{f}$  belong to appropriate Triebel-Lizorkin spaces (in which case the conormal  $\partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}}$  belongs to a suitable diagonal boundary Besov space).

**Remark.** Since the conormal  $\partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}}$  has been defined for a class of (triplets of) functions  $\vec{u}, \pi, \vec{f}$  for which the expression  $\left[ (\nabla \vec{u})^\top + \lambda(\nabla \vec{u}) \right] \nu - \pi \nu$  is, in the standard sense of the trace theory, utterly ill-defined on  $\partial\Omega$ , it is appropriate to remark that  $(\vec{u}, \pi, \vec{f}) \mapsto \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}}$  is not an extension of the operation  $(\vec{u}, \pi, ) \mapsto \text{Tr} \left[ (\nabla \vec{u})^\top + \lambda(\nabla \vec{u}) \right] \nu - \text{Tr} \pi \nu$  in an ordinary sense. In fact, it is more appropriate to regard the former as a “renormalization” of the latter trace, in a fashion that depends strongly on the choice of  $\vec{f} \in A_{s+1/p-2,0}^{p,q}(\Omega)^n$  as an extension of  $\Delta \vec{u} - \nabla \pi \in A_{s+1/p-2,z}^{p,q}(\Omega)^n$ .

To further shed light on this issue, recall that, for  $\vec{u} \in L_1^2(\Omega)^n$ ,  $\pi \in L^2(\Omega)$ ,  $\Delta \vec{u} - \nabla \pi$  is naturally defined as a linear functional in  $(L_{1,0}^2(\Omega)^n)^*$ . The choice of  $\vec{f}$  is the choice of an extension of this linear functional to a functional in  $(L_1^2(\Omega)^n)^* = L_{-1,0}^2(\Omega)^n$ . As an example, consider  $\vec{u} \in L_1^2(\Omega)^n$ ,  $\pi \in L^2(\Omega)$ , and suppose that actually  $\vec{u} \in L_2^2(\Omega)$ ,  $\pi \in L_1^2(\Omega)$  so  $\text{Tr} \left[ (\nabla \vec{u})^\top + \lambda(\nabla \vec{u}) \right] \nu - \text{Tr} \pi \nu$  is well defined in  $L^2(\partial\Omega)^n$ . In this case,  $\Delta \vec{u} - \nabla \pi \in L^2(\Omega)^n$  has a “natural” extension  $\vec{f}_0 \in L_{-1,0}^2(\Omega)^n$  (i.e.,  $\vec{f}_0$  is the extension of  $\Delta \vec{u} - \nabla \pi$  to  $\mathbb{R}^n$  by setting this equal zero outside  $\Omega$ ). Any other extension  $\vec{f}_1 \in L_{-1,0}^2(\Omega)^n$  differs from  $\vec{f}_0$  by a distribution  $\vec{\eta} \in L_{-1}^2(\mathbb{R}^n)^n$  supported on  $\partial\Omega$ . As is well-known, the space of such distributions is nontrivial. In fact, we have

$$(3.21) \quad \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}_0} = \text{Tr} \left[ (\nabla \vec{u})^\top + \lambda(\nabla \vec{u}) \right] \nu - \text{Tr} \pi \nu \quad \text{in } L^2(\partial\Omega)^n,$$

but if  $\vec{\eta} \neq 0$  then  $\partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}_0}$  is not equal to  $\partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}_1}$ . Indeed, by linearity we have that  $\partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}_1} = \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}_0} + \partial_\nu^\lambda(\vec{0}, 0)_{\vec{\eta}}$  and (3.20) shows that

$$(3.22) \quad \left\langle \partial_\nu^\lambda(\vec{0}, 0)_{\vec{\eta}}, \vec{\psi} \right\rangle = \left\langle \vec{\eta}, \text{Ex}(\vec{\psi}) \right\rangle$$

for every  $\vec{\psi} \in L_{1/2}^2(\partial\Omega)^n$ . Consequently,  $\partial_\nu^\lambda(\vec{0}, 0)_{\vec{\eta}} \neq 0$  if  $\vec{\eta} \neq 0$ .

We continue by registering an natural integration by parts formula, which builds on the definition of the “renormalized” conormal (3.20).

**Proposition 3.5.** *Assume that  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain. Fix  $s \in (0, 1)$ , as well as  $1 < p, q < \infty$ , and denote by  $p', q'$  the Hölder conjugate exponents of  $p$  and  $q$ , respectively.*

*Next, suppose that  $\vec{w} \in A_{1-s+1/p'}^{p',q'}(\Omega)^n$ ,  $\vec{u} \in A_{s+\frac{1}{p}}^{p,q}(\Omega)^n$ ,  $\pi \in A_{s+\frac{1}{p}-1}^{p,q}(\Omega)$  and  $\vec{f} \in A_{s+\frac{1}{p}-2,0}^{p,q}(\Omega)^n$  are such that  $\Delta \vec{u} - \nabla \pi = \vec{f}|_\Omega$  in  $\Omega$  (where, as usual,  $A \in \{B, F\}$ ). Then, for every  $\lambda \in \mathbb{R}$ , the following integration by parts formula holds:*

$$(3.23) \quad \left\langle \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}}, \text{Tr} \vec{w} \right\rangle = \left\langle \vec{f}, \vec{w} \right\rangle + \mathbb{A}_\lambda \left( \nabla \vec{u}, \nabla \vec{w} \right) - \left\langle \pi, \text{div} \vec{w} \right\rangle.$$

*Proof.* By linearity, it suffices to show that

$$(3.24) \quad \langle \vec{f}, \vec{w} \rangle + \mathbb{A}_\lambda(\nabla \vec{u}, \nabla \vec{w}) - \langle \pi, \operatorname{div} \vec{w} \rangle = 0$$

if  $\vec{w}, \vec{u}, \pi, \vec{f}$  are as in the statement of the proposition and, in addition,  $\operatorname{Tr} \vec{w} = 0$ . Note that the latter condition entails that  $\vec{w} \in A_{1-s,z}^{p',q'}(\Omega)^n$  by (3.12), (3.16). Thus, by (3.13), (3.17),  $\vec{w}$  can be approximated in  $A_{1-s,z}^{p',q'}(\Omega)^n$  by a sequence of vector fields  $\vec{w}_j \in C_c^\infty(\Omega)^n$ . Since, thanks to the fact that  $\Delta \vec{u} - \nabla \pi = \vec{f}|_\Omega$  as distributions in  $\Omega$ , we have

$$(3.25) \quad \langle \vec{f}, \vec{w}_j \rangle + \mathbb{A}_\lambda(\nabla \vec{u}, \nabla \vec{w}_j) - \langle \pi, \operatorname{div} \vec{w}_j \rangle = 0, \quad j \in \mathbb{N},$$

we can obtain (3.24) by letting  $j \rightarrow \infty$ .  $\square$

In order to continue, we introduce the scales

$$(3.26) \quad \mathcal{B}_s^{p,q}(\Omega) := \left\{ (\vec{u}, \pi, \vec{f}) \in B_{s+\frac{1}{p}}^{p,q}(\Omega) \oplus B_{s+\frac{1}{p}-1}^{p,q}(\Omega) \oplus B_{s+\frac{1}{p}-2,0}^{p,q}(\Omega) : \Delta \vec{u} - \nabla \pi = \vec{f}|_\Omega \right\},$$

and

$$(3.27) \quad \mathcal{F}_s^{p,q}(\Omega) := \left\{ (\vec{u}, \pi, \vec{f}) \in F_{s+\frac{1}{p}}^{p,q}(\Omega) \oplus F_{s+\frac{1}{p}-1}^{p,q}(\Omega) \oplus F_{s+\frac{1}{p}-2,0}^{p,q}(\Omega) : \Delta \vec{u} - \nabla \pi = \vec{f}|_\Omega \right\}.$$

**Corollary 3.6.** *Suppose that  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain, and assume that  $s \in (0, 1)$ ,  $1 < p, q < \infty$ ,  $1/p + 1/p' = 1/q + 1/q' = 1$ . Then*

$$(3.28) \quad \begin{aligned} \langle \vec{f}, \vec{w} \rangle - \langle \vec{g}, \vec{u} \rangle &= \left\langle \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}}, \operatorname{Tr} \vec{w} \right\rangle - \left\langle \partial_\nu^\lambda(\vec{w}, \rho)_{\vec{g}}, \operatorname{Tr} \vec{u} \right\rangle \\ &+ \left\langle \pi, \operatorname{div} \vec{w} \right\rangle - \left\langle \rho, \operatorname{div} \vec{u} \right\rangle \end{aligned}$$

provided either

$$(3.29) \quad (\vec{u}, \pi, \vec{f}) \in \mathcal{B}_s^{p,q}(\Omega), \quad (\vec{w}, \rho, \vec{g}) \in \mathcal{B}_{1-s}^{p',q'}(\Omega),$$

or

$$(3.30) \quad (\vec{u}, \pi, \vec{f}) \in \mathcal{F}_s^{p,q}(\Omega), \quad (\vec{w}, \rho, \vec{g}) \in \mathcal{F}_{1-s}^{p',q'}(\Omega).$$

*Proof.* This follows from (3.23) much as (2.35) follows from (2.34).  $\square$

#### 4. SESQUILINEAR FORMS AND THEIR ASSOCIATED OPERATORS

In this section we describe a few basic facts on sesquilinear forms and linear operators associated with them. Throughout, given two Banach spaces  $\mathcal{X}, \mathcal{Y}$ , we denote by  $\mathcal{B}(\mathcal{X}, \mathcal{Y})$  the space of linear, bounded operators from  $\mathcal{X}$  into  $\mathcal{Y}$ , equipped with the strong operator norm. Also, we let  $I_{\mathcal{X}}$  stand for the identity operator on  $\mathcal{X}$ . Finally, we adopt the convention that if  $\mathcal{X}$  is a Banach space then  $\mathcal{X}^*$  denotes the *adjoint space* of continuous conjugate linear functionals on  $\mathcal{X}$ , also known as the *conjugate dual* of  $\mathcal{X}$ . In this scenario, we let  $\mathcal{X}\langle \cdot, \cdot \rangle_{\mathcal{X}^*}$  denote the duality pairing between  $\mathcal{X}$  and  $\mathcal{X}^*$ .

Let  $\mathcal{H}$  be a complex separable Hilbert space with scalar product  $(\cdot, \cdot)_{\mathcal{H}}$  (antilinear in the first and linear in the second argument),  $\mathcal{V}$  a reflexive Banach space continuously and densely embedded into  $\mathcal{H}$ . Then also  $\mathcal{H}$  embeds continuously and densely into  $\mathcal{V}^*$ , i.e.,

$$(4.1) \quad \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^* \quad \text{continuously and densely.}$$

Here the continuous embedding  $\mathcal{H} \hookrightarrow \mathcal{V}^*$  is accomplished via the identification

$$(4.2) \quad \mathcal{H} \ni u \mapsto (\cdot, u)_{\mathcal{H}} \in \mathcal{V}^*.$$

In particular, if the sesquilinear form

$$(4.3) \quad \mathcal{V}\langle \cdot, \cdot \rangle_{\mathcal{V}^*}: \mathcal{V} \times \mathcal{V}^* \rightarrow \mathbb{C}$$

denotes the duality pairing between  $\mathcal{V}$  and  $\mathcal{V}^*$ , then

$$(4.4) \quad \mathcal{V}\langle u, v \rangle_{\mathcal{V}^*} = (u, v)_{\mathcal{H}}, \quad u \in \mathcal{V}, \quad v \in \mathcal{H} \hookrightarrow \mathcal{V}^*,$$

that is, the  $\mathcal{V}, \mathcal{V}^*$  pairing  $\mathcal{V}\langle \cdot, \cdot \rangle_{\mathcal{V}^*}$  is compatible with the scalar product  $(\cdot, \cdot)_{\mathcal{H}}$  in  $\mathcal{H}$ .

Let  $T \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*)$ . Since  $\mathcal{V}$  is reflexive, i.e.  $(\mathcal{V}^*)^* = \mathcal{V}$ , one has

$$(4.5) \quad T: \mathcal{V} \rightarrow \mathcal{V}^*, \quad T^*: \mathcal{V} \rightarrow \mathcal{V}^*$$

and

$$(4.6) \quad \mathcal{V}\langle u, Tv \rangle_{\mathcal{V}^*} = \mathcal{V}^*\langle T^*u, v \rangle_{(\mathcal{V}^*)^*} = \mathcal{V}^*\langle T^*u, v \rangle_{\mathcal{V}} = \overline{\mathcal{V}\langle v, T^*u \rangle_{\mathcal{V}^*}}.$$

*Self-adjointness* of  $T$  is then defined as the property that  $T = T^*$ , that is,

$$(4.7) \quad \mathcal{V}\langle u, Tv \rangle_{\mathcal{V}^*} = \mathcal{V}^*\langle Tu, v \rangle_{\mathcal{V}} = \overline{\mathcal{V}\langle v, Tu \rangle_{\mathcal{V}^*}}, \quad u, v \in \mathcal{V},$$

*nonnegativity* of  $T$  is defined as the demand that

$$(4.8) \quad \mathcal{V}\langle u, Tu \rangle_{\mathcal{V}^*} \geq 0, \quad u \in \mathcal{V},$$

and *boundedness from below* of  $T$  by  $c \in \mathbb{R}$  is defined as the property that

$$(4.9) \quad \mathcal{V}\langle u, Tu \rangle_{\mathcal{V}^*} \geq c \|u\|_{\mathcal{H}}^2, \quad \forall u \in \mathcal{V}.$$

(Note that, by (4.4), this is equivalent to  $\mathcal{V}\langle u, Tu \rangle_{\mathcal{V}^*} \geq c \mathcal{V}\langle u, u \rangle_{\mathcal{V}^*}$  for all  $u \in \mathcal{V}$ .)

Next, let the sesquilinear form  $a(\cdot, \cdot): \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$  (antilinear in the first and linear in the second argument) be  $\mathcal{V}$ -*bounded*. That is, there exists a  $c_a > 0$  such that

$$(4.10) \quad |a(u, v)| \leq c_a \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}, \quad u, v \in \mathcal{V}.$$

Then  $\tilde{A}$  defined by

$$(4.11) \quad \tilde{A}: \begin{cases} \mathcal{V} \rightarrow \mathcal{V}^*, \\ v \mapsto \tilde{A}v = a(\cdot, v), \end{cases}$$

satisfies

$$(4.12) \quad \tilde{A} \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*) \text{ and } {}_{\mathcal{V}}\langle u, \tilde{A}v \rangle_{\mathcal{V}^*} = a(u, v), \quad u, v \in \mathcal{V}.$$

In the sequel, we shall refer to  $\tilde{A}$  as *the operator induced by the form*  $a(\cdot, \cdot)$ .

Assuming further that  $a(\cdot, \cdot)$  is *symmetric*, that is,

$$(4.13) \quad a(u, v) = \overline{a(v, u)}, \quad u, v \in \mathcal{V},$$

and that  $a$  is  $\mathcal{V}$ -*coercive*, that is, there exists a constant  $C_0 > 0$  such that

$$(4.14) \quad a(u, u) \geq C_0 \|u\|_{\mathcal{V}}^2, \quad u \in \mathcal{V},$$

respectively, then,

$$(4.15) \quad \tilde{A}: \mathcal{V} \rightarrow \mathcal{V}^* \text{ is bounded, self-adjoint, and boundedly invertible.}$$

Moreover, denoting by  $A$  the part of  $\tilde{A}$  in  $\mathcal{H}$ , defined by

$$(4.16) \quad D(A) := \{u \in \mathcal{V} : \tilde{A}u \in \mathcal{H}\} \subseteq \mathcal{H}, \quad A := \tilde{A}|_{D(A)}: D(A) \rightarrow \mathcal{H},$$

then  $A$  is a (possibly unbounded) self-adjoint operator in  $\mathcal{H}$  satisfying

$$(4.17) \quad A \geq C_0 I_{\mathcal{H}},$$

$$(4.18) \quad D(A^{1/2}) = \mathcal{V}.$$

In particular,

$$(4.19) \quad A^{-1} \in \mathcal{B}(\mathcal{H}).$$

The facts (4.1)–(4.19) are a consequence of the Lax–Milgram theorem and the second representation theorem for symmetric sesquilinear forms. Details can be found, for instance, in [2, §VI.3, §VII.1], [6, Ch. IV], and [17].

Next, consider a symmetric form  $b(\cdot, \cdot): \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$  and assume that  $b$  is *bounded from below* by  $c_b \in \mathbb{R}$ , that is,

$$(4.20) \quad b(u, u) \geq c_b \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{V}.$$

Introducing the scalar product  $(\cdot, \cdot)_{\mathcal{V}(b)}: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$  (with associated norm  $\|\cdot\|_{\mathcal{V}(b)}$ ) by

$$(4.21) \quad (u, v)_{\mathcal{V}(b)} := b(u, v) + (1 - c_b)(u, v)_{\mathcal{H}}, \quad u, v \in \mathcal{V},$$

turns  $\mathcal{V}$  into a pre-Hilbert space  $(\mathcal{V}; (\cdot, \cdot)_{\mathcal{V}(b)})$ , which we denote by  $\mathcal{V}(b)$ . The form  $b$  is called *closed* if  $\mathcal{V}(b)$  is actually complete, and hence a Hilbert space. The form  $b$  is called *closable* if it has a closed extension. If  $b$  is closed, then

$$(4.22) \quad |b(u, v) + (1 - c_b)(u, v)_{\mathcal{H}}| \leq \|u\|_{\mathcal{V}(b)} \|v\|_{\mathcal{V}(b)}, \quad u, v \in \mathcal{V},$$



and

$$(4.23) \quad |b(u, u) + (1 - c_b)\|u\|_{\mathcal{H}}^2| = \|u\|_{\mathcal{V}(b)}^2, \quad u \in \mathcal{V},$$

show that the form  $b(\cdot, \cdot) + (1 - c_b)(\cdot, \cdot)_{\mathcal{H}}$  is a symmetric,  $\mathcal{V}$ -bounded, and  $\mathcal{V}$ -coercive sesquilinear form.

Hence, by (4.11) and (4.12), there exists a linear map

$$(4.24) \quad \tilde{B}_{c_b}: \begin{cases} \mathcal{V}(b) \rightarrow \mathcal{V}(b)^*, \\ v \mapsto \tilde{B}_{c_b}v := b(\cdot, v) + (1 - c_b)(\cdot, v)_{\mathcal{H}}, \end{cases}$$

with

$$(4.25) \quad \tilde{B}_{c_b} \in \mathcal{B}(\mathcal{V}(b), \mathcal{V}(b)^*) \text{ and } {}_{\mathcal{V}(b)}\langle u, \tilde{B}_{c_b}v \rangle_{\mathcal{V}(b)^*} = b(u, v) + (1 - c_b)(u, v)_{\mathcal{H}}, \quad u, v \in \mathcal{V}.$$

Introducing the linear map

$$(4.26) \quad \tilde{B} := \tilde{B}_{c_b} + (c_b - 1)\tilde{I}: \mathcal{V}(b) \rightarrow \mathcal{V}(b)^*,$$

where  $\tilde{I}: \mathcal{V}(b) \hookrightarrow \mathcal{V}(b)^*$  denotes the continuous inclusion (embedding) map of  $\mathcal{V}(b)$  into  $\mathcal{V}(b)^*$ , one obtains a self-adjoint operator  $B$  in  $\mathcal{H}$  by restricting  $\tilde{B}$  to  $\mathcal{H}$ ,

$$(4.27) \quad D(B) = \{u \in \mathcal{V} : \tilde{B}u \in \mathcal{H}\} \subseteq \mathcal{H}, \quad B = \tilde{B}|_{D(B)}: D(B) \rightarrow \mathcal{H},$$

satisfying the following properties:

$$(4.28) \quad B \geq c_b I_{\mathcal{H}},$$

$$(4.29) \quad D(|B|^{1/2}) = D((B - c_b I_{\mathcal{H}})^{1/2}) = \mathcal{V},$$

$$(4.30) \quad b(u, v) = (|B|^{1/2}u, U_B|B|^{1/2}v)_{\mathcal{H}}$$

$$(4.31) \quad = ((B - c_b I_{\mathcal{H}})^{1/2}u, (B - c_b I_{\mathcal{H}})^{1/2}v)_{\mathcal{H}} + c_b(u, v)_{\mathcal{H}}$$

$$(4.32) \quad = {}_{\mathcal{V}(b)}\langle u, \tilde{B}v \rangle_{\mathcal{V}(b)^*}, \quad u, v \in \mathcal{V},$$

$$(4.33) \quad b(u, v) = (u, Bv)_{\mathcal{H}}, \quad u \in \mathcal{V}, v \in D(B),$$

$$D(B) = \{v \in \mathcal{V} : \text{there exists } f_v \in \mathcal{H} \text{ such that}$$

$$(4.34) \quad b(w, v) = (w, f_v)_{\mathcal{H}} \text{ for all } w \in \mathcal{V}\},$$

$$Bu = f_u, \quad u \in D(B),$$

$$(4.35) \quad D(B) \text{ is dense in } \mathcal{H} \text{ and in } \mathcal{V}(b).$$

Properties (4.34) and (4.35) uniquely determine  $B$ . Here  $U_B$  in (4.31) is the partial isometry in the polar decomposition of  $B$ , that is,

$$(4.36) \quad B = U_B|B|, \quad |B| = (B^*B)^{1/2}.$$

**Definition 4.1.** The operator  $B$  is called the operator associated with the form  $b(\cdot, \cdot)$ .

The norm in the Hilbert space  $\mathcal{V}(b)^*$  is given by

$$(4.37) \quad \|\ell\|_{\mathcal{V}(b)^*} = \sup\{|\mathcal{V}(b)\langle u, \ell \rangle_{\mathcal{V}(b)^*}| : \|u\|_{\mathcal{V}(b)} \leq 1\}, \quad \ell \in \mathcal{V}(b)^*,$$

with associated scalar product,

$$(4.38) \quad (\ell_1, \ell_2)_{\mathcal{V}(b)^*} = \mathcal{V}(b)\langle (\tilde{B} + (1 - c_b)\tilde{I})^{-1}\ell_1, \ell_2 \rangle_{\mathcal{V}(b)^*}, \quad \ell_1, \ell_2 \in \mathcal{V}(b)^*.$$

Since

$$(4.39) \quad \|(\tilde{B} + (1 - c_b)\tilde{I})v\|_{\mathcal{V}(b)^*} = \|v\|_{\mathcal{V}(b)}, \quad v \in \mathcal{V},$$

the Riesz representation theorem yields

$$(4.40) \quad (\tilde{B} + (1 - c_b)\tilde{I}) \in \mathcal{B}(\mathcal{V}(b), \mathcal{V}(b)^*) \text{ and } (\tilde{B} + (1 - c_b)\tilde{I}) : \mathcal{V}(b) \rightarrow \mathcal{V}(b)^* \text{ is unitary.}$$

In addition,

$$(4.41) \quad \begin{aligned} \mathcal{V}(b)\langle u, (\tilde{B} + (1 - c_b)\tilde{I})v \rangle_{\mathcal{V}(b)^*} &= ((B + (1 - c_b)I_{\mathcal{H}})^{1/2}u, (B + (1 - c_b)I_{\mathcal{H}})^{1/2}v)_{\mathcal{H}} \\ &= (u, v)_{\mathcal{V}(b)}, \quad u, v \in \mathcal{V}(b). \end{aligned}$$

In particular,

$$(4.42) \quad \|(B + (1 - c_b)I_{\mathcal{H}})^{1/2}u\|_{\mathcal{H}} = \|u\|_{\mathcal{V}(b)}, \quad u \in \mathcal{V}(b),$$

and hence

$$(4.43) \quad (B + (1 - c_b)I_{\mathcal{H}})^{1/2} \in \mathcal{B}(\mathcal{V}(b), \mathcal{H}) \text{ and } (B + (1 - c_b)I_{\mathcal{H}})^{1/2} : \mathcal{V}(b) \rightarrow \mathcal{H} \text{ is unitary.}$$

The facts (4.20)–(4.43) comprise the second representation theorem of sesquilinear forms (cf. [6, Sect. IV.2], [9, Sects. 1.2–1.5], and [15, Sect. VI.2.6]).

## 5. FRACTIONAL POWERS AND SEMIGROUP THEORY

Assume that  $\mathcal{H}$  is a (possibly complex) separable Hilbert space with scalar product  $(\cdot, \cdot)_{\mathcal{H}}$  and that  $\mathcal{V}$  a reflexive Banach space continuously and densely embedded into  $\mathcal{H}$ . Also, fix a sesquilinear form  $b(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ , which is assumed to be symmetric, nonnegative, bounded, and which satisfies the following coercivity condition: There exist  $C_0 \in \mathbb{R}$  and  $C_1 > 0$  such that

$$(5.1) \quad b(u, u) + C_0\|u\|_{\mathcal{H}}^2 \geq C_1\|u\|_{\mathcal{V}}^2, \quad u \in \mathcal{V}.$$

As a consequence,  $\|\cdot\|_{\mathcal{V}(b)} \approx \|\cdot\|_{\mathcal{V}}$ . Thus  $\mathcal{V}(b) = \mathcal{V}$  and, hence,  $b(\cdot, \cdot)$  is also closed.

Let  $B : D(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$  be the (possibly unbounded) operator associated with the form  $b(\cdot, \cdot)$  as in Definition 4.1. In particular,  $B$  is self-adjoint and nonnegative. Also,  $tI_{\mathcal{H}} + B$  is invertible on  $\mathcal{H}$  for every  $t > 0$ , and  $\|t(I_{\mathcal{H}} + B)^{-1}\|_{\mathcal{B}(\mathcal{H}, \mathcal{H})} \leq C$  for  $t > 0$  (cf., e.g., Proposition 1.22 on p. 13 in [24]). In fact, there exist  $\theta \in (0, \pi/2)$  and a finite constant

$C > 0$  such that  $\Sigma_\theta := \{z \in \mathbb{C} : |\arg(z - 1)| \leq \theta + \pi/2\}$  is contained in  $\mathbb{C} \setminus \text{Spec}(B)$  (where  $\text{Spec}(B)$  denotes the spectrum of  $B$  as an operator on  $\mathcal{H}$ ) and

$$(5.2) \quad \|(zI_{\mathcal{H}} + B)^{-1}\|_{\mathcal{B}(\mathcal{H}, \mathcal{H})} \leq \frac{C}{1 + |z|}, \quad z \in \Sigma_\theta,$$

i.e.,  $B$  is *sectorial*. See, e.g., Theorem 3 on p.374 and Proposition 3 on p.380 in [3]. In particular, the operator  $B$  generates an analytic semigroup on  $\mathcal{H}$  according to the formula

$$(5.3) \quad e^{zB}u := \frac{1}{2\pi i} \int_{\Gamma_{\theta'}} e^{-\zeta z} (\zeta I_{\mathcal{H}} + B)^{-1} u d\zeta, \quad |\arg(z)| < \pi/2 - \theta', \quad u \in \mathcal{H},$$

where  $\theta' \in (\theta, \pi/2)$  and  $\Gamma_{\theta'} := \{\pm r e^{i\theta'} : r > 0\}$ . Cf. [3] and [25] for a more detailed discussion in this regard.

Moving on, we denote by  $\{E_B(\mu)\}_{\mu \in \mathbb{R}}$  the family of spectral projections associated with  $B$ , and for each  $u \in \mathcal{H}$  introduce the function  $\rho_u$  by

$$(5.4) \quad \rho_u : \mathbb{R} \longrightarrow [0, \infty), \quad \rho_u(\mu) := (E_B(\mu)u, u)_{\mathcal{H}} = \|E_B(\mu)u\|_{\mathcal{H}}^2.$$

Clearly,  $\rho_u$  is bounded, non-decreasing, right-continuous, and

$$(5.5) \quad \lim_{\mu \downarrow -\infty} \rho_u(\mu) = 0, \quad \lim_{\mu \uparrow \infty} \rho_u(\mu) = \|u\|_{\mathcal{H}}^2, \quad \forall u \in \mathcal{H}.$$

Hence,  $\rho_u$  generates a measure, denoted by  $d\rho_u$ , in a canonical manner. A function  $f: \mathbb{R} \rightarrow \mathbb{C}$  is then called *dE<sub>B</sub>-measurable* if it is  $d\rho_u$ -measurable for each  $u \in \mathcal{H}$ . As is well-known, all Borel measurable functions are  $dE_B$ -measurable functions. For a Borel measurable function  $f: \mathbb{R} \rightarrow \mathbb{C}$  we then define the (possibly) unbounded operator by setting

$$(5.6) \quad \begin{aligned} D(f(B)) &:= \left\{ u \in \mathcal{H} : \int_{\mathbb{R}} |f| d\rho_u < +\infty \right\} \\ f(B)u &:= \int_{\mathbb{R}} f(\mu) dE_B(\mu)u, \quad u \in D(f(B)). \end{aligned}$$

In particular, for each  $\alpha \in [0, 1]$ , the fractional power  $B^\alpha$  of  $B$  is a self-adjoint operator

$$(5.7) \quad B^\alpha : D(B^\alpha) \subset \mathcal{H} \longrightarrow \mathcal{H}.$$

Since in our case  $B$  is maximally accretive, then so is  $B^\alpha$  if  $\alpha \in (0, 1)$  and for every  $u \in D(B) \subset D(B^\alpha)$  we have the representation

$$(5.8) \quad B^\alpha u = \frac{\sin(\pi z)}{\pi} \int_0^\infty t^\alpha B(tI_{\mathcal{H}} + B)^{-1} u \frac{dt}{t}.$$

See [13], [15]. Other properties are discussed in, e.g., Pazy's book [25], to which we refer the interested reader. Here we only wish to summarize some well-known results of T. Kato and J.-L. Lions (see [14], [17]) which are relevant for our work. Specifically, if  $B$  is as above, then

$$(5.9) \quad D(B^{1/2}) = \mathcal{V}$$

and, with  $[\cdot, \cdot]_\theta$  denoting the complex interpolation bracket,

$$(5.10) \quad D(B^\theta) = [\mathcal{H}, D(B)]_\theta, \quad 0 \leq \theta \leq 1.$$

Hence, by the reiteration theorem for the complex method, the family

$$(5.11) \quad \left\{ D(B^{\frac{s}{2}}) : 0 \leq s \leq 2 \right\} \quad \text{is a complex interpolation scale.}$$

In particular,

$$(5.12) \quad D(B^{\theta/2}) = [\mathcal{H}, \mathcal{V}]_\theta, \quad 0 \leq \theta \leq 1.$$

We wish to further elaborate on this topic by shedding some light on the nature of  $D(B^\alpha)$  when  $\alpha \in (1/2, 1)$ . This requires some preparations. To get started, denote by  $\tilde{B} \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*)$  the operator induced by the form  $b(\cdot, \cdot)$  (so that  $B$  is the part of  $\tilde{B}$  in  $\mathcal{H}$ ), and let  $\tilde{I}$  stand for the inclusion of  $\mathcal{V}$  into  $\mathcal{V}^*$ . It then follows from (5.1) that

$$(5.13) \quad (\tilde{I} + \tilde{B}) \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*) \quad \text{is an isomorphism.}$$

The idea is to find another suitable context in which the operator  $\tilde{I} + \tilde{B}$  is an isomorphism, and then interpolate between this and (5.13). However, in contrast to what goes on for boundedness, invertibility is not, generally speaking, preserved under interpolation. There are, nonetheless, certain specific settings in which this is true. To discuss such a case recall that, if  $(X_0, X_1)$  are a couple of compatible Banach spaces,  $X_0 \cap X_1$  and  $X_0 + X_1$  are equipped, respectively, with the norms

$$(5.14) \quad \begin{aligned} \|x\|_{X_0 \cap X_1} &:= \max \{ \|x\|_{X_0}, \|x\|_{X_1} \}, & \text{and} \\ \|z\|_{X_0 + X_1} &= \inf \{ \|x_0\|_{X_0} + \|x_1\|_{X_1} : z = x_0 + x_1, \ x_i \in X_i, \ i = 0, 1 \}. \end{aligned}$$

We have:

**Lemma 5.1.** *Let  $(X_0, X_1)$  and  $(Y_0, Y_1)$  be two couples of compatible Banach spaces and assume that  $T : X_0 + X_1 \longrightarrow Y_0 + Y_1$  is a linear operator with the property that*

$$(5.15) \quad T : X_i \longrightarrow Y_i \quad \text{is an isomorphism, } i = 0, 1.$$

*In addition, assume that there exist Banach spaces  $X', Y'$  such that the inclusions*

$$(5.16) \quad X' \hookrightarrow X_0 \cap X_1, \quad Y' \hookrightarrow Y_0 \cap Y_1,$$

*are continuous with dense range, and that*

$$(5.17) \quad T : X' \longrightarrow Y' \quad \text{is an isomorphism.}$$

*Then the operator*

$$(5.18) \quad T : [X_0, X_1]_\theta \longrightarrow [Y_0, Y_1]_\theta$$

*is an isomorphism for each  $0 \leq \theta \leq 1$ .*

*Proof.* Denote by  $R_i \in \mathcal{B}(Y_i, X_i)$ ,  $i = 0, 1$ , the inverses of  $T$  in (5.15). Since the operators  $R_0$  and  $R_1$  coincide as mappings in  $\mathcal{B}(Y', X')$ , by density they also agree as mappings in  $\mathcal{B}(Y_0 \cap Y_1, X_0 \cap X_1)$ . It is therefore meaningful to define

$$(5.19) \quad \begin{aligned} R : Y_0 + Y_1 &\longrightarrow X_0 + X_1, \quad \text{by} \\ R(y_0 + y_1) &:= R_0(y_0) + R_1(y_1), \quad y_i \in Y_i, \quad i = 0, 1. \end{aligned}$$

Then  $R$  is a linear operator which belongs to  $\mathcal{B}(Y_0, X_0) \cap \mathcal{B}(Y_1, X_1)$ . Thus, by the interpolation property,  $R$  maps  $[Y_0, Y_1]_\theta$  boundedly into  $[X_0, X_1]_\theta$  for every  $\theta \in [0, 1]$ . In this latter context,  $R$  provides an inverse for  $T : [X_0, X_1]_\theta \longrightarrow [Y_0, Y_1]_\theta$ , since  $RT = I_{X_0 \cap X_1}$  on  $X_0 \cap X_1$ , which is a dense subspace of  $[X_0, X_1]_\theta$ , and  $TR = I_{Y_0 \cap Y_1}$  on  $Y_0 \cap Y_1$ , which is a dense subspace of  $[Y_0, Y_1]_\theta$ . This proves that the operator in (5.18) is indeed an isomorphism for every  $\theta \in [0, 1]$ .  $\square$

After this preamble, we are ready to present the following.

**Proposition 5.2.** *With the above assumptions and notation,*

$$(5.20) \quad D(B^{\frac{1+\theta}{2}}) = (\tilde{I} + \tilde{B})^{-1} \left( D(B^{\frac{1-\theta}{2}}) \right)^*$$

for every  $0 \leq \theta \leq 1$ .

*Proof.* As already remarked above, the operator  $\tilde{I} + \tilde{B} : \mathcal{V} \rightarrow \mathcal{V}^*$  is boundedly invertible. We claim that

$$(5.21) \quad \tilde{I} + \tilde{B} : D(B) \longrightarrow \mathcal{H}$$

is invertible as well, when  $D(B)$  is equipped with the graph norm  $u \mapsto \|u\|_{\mathcal{H}} + \|Bu\|_{\mathcal{H}}$ . Indeed, this operator is clearly well-defined, linear and bounded, since  $\tilde{B}$  coincides with  $B$  on  $D(B)$ . Also, the fact that the operator in (5.13) is one-to-one readily entails that so is (5.21). To see that the operator (5.21) is onto, pick an arbitrary  $w \in \mathcal{H} \hookrightarrow \mathcal{V}^*$ . It follows from (5.13) that there exists  $u \in \mathcal{V} \hookrightarrow \mathcal{H}$  such that  $(\tilde{I} + \tilde{B})u = w$ . In turn, this implies that  $\tilde{B}u = w - u \in \mathcal{H}$  and, hence,  $u \in D(B)$ . This shows that the operator (5.21) is onto, hence ultimately invertible.

Interpolating between (5.13) and (5.21) then proves (with the help of Lemma 5.1, (5.9)-(5.10), and the duality theorem for the complex method) that the operator

$$(5.22) \quad \tilde{I} + \tilde{B} : D(B^{\frac{1+\theta}{2}}) = [\mathcal{V}, D(B)]_\theta \rightarrow [\mathcal{V}^*, \mathcal{H}]_\theta = \left( [\mathcal{H}, \mathcal{V}]_{1-\theta} \right)^* = \left( D(B^{\frac{1-\theta}{2}}) \right)^*$$

is an isomorphism, for every  $0 \leq \theta \leq 1$ . From this, (5.20) readily follows.  $\square$

## 6. THE DEFINITION OF THE NEUMANN-STOKES OPERATOR

In this section we define the Stokes operator when equipped with Neumann boundary conditions in Lipschitz domains in  $\mathbb{R}^n$ . Subsequently, in Theorem 6.7, we study the functional analytic properties of this operator. We begin by making the following:

**Definition 6.1.** Let  $\Omega \subset \mathbb{R}^n$  be a Lipschitz domain and assume that  $\lambda \in \mathbb{R}$  is fixed. Define the Stokes operator with Neumann boundary condition as the unbounded operator

$$(6.1) \quad B_\lambda : D(B_\lambda) \subset \mathcal{H} \longrightarrow \mathcal{H}$$

with domain

$$(6.2) \quad D(B_\lambda) := \left\{ \vec{u} \in \mathcal{V} : \text{there exists } \pi \in L^2(\Omega) \text{ so that } \vec{f} := -\Delta \vec{u} + \nabla \pi \in \mathcal{H} \right. \\ \left. \text{and such that } \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}} = 0 \text{ in } L^2_{-1/2}(\partial\Omega)^n \right\},$$

and acting according to

$$(6.3) \quad B_\lambda \vec{u} := -\Delta \vec{u} + \nabla \pi, \quad \vec{u} \in D(B_\lambda),$$

assuming that the pair  $(\vec{u}, \pi)$  satisfies the requirements in the definition of  $D(B_\lambda)$ .

As it stands, it is not entirely obvious that the above definition is indeed coherent and our first order of business is to clarify this issue. We do so in a series of lemmas, starting with:

**Lemma 6.2.** *If the pair  $(\vec{u}, \pi)$  satisfies the requirements in the definition of  $D(B_\lambda)$ , then  $\Delta \pi = 0$  in  $\Omega$ .*

*Proof.* Since the vector fields  $\vec{u}$  and  $\vec{f} := -\Delta \vec{u} - \nabla \pi$  are both divergence-free, it follows that  $\Delta \pi = \operatorname{div}(-\Delta \vec{u} - \nabla \pi) = \operatorname{div} \vec{f} = 0$ .  $\square$

**Lemma 6.3.** *If  $\vec{u} \in D(B_\lambda)$ , then there exists a unique scalar function  $\pi \in L^2(\Omega)$  such that  $\vec{f} := -\Delta \vec{u} - \nabla \pi \in \mathcal{H}$  and  $\partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}} = 0$  in  $L^2_{-1/2}(\partial\Omega)^n$ .*

*Proof.* Fix a vector field  $\vec{u} \in D(B_\lambda)$  and assume that  $\pi_j \in L^2(\Omega)$ ,  $j = 1, 2$ , are such that

$$(6.4) \quad \vec{f}_j := -\Delta \vec{u} - \nabla \pi_j \in \mathcal{H} \quad \text{and} \quad \partial_\nu^\lambda(\vec{u}, \pi_j)_{\vec{f}_j} = 0 \quad \text{in } L^2_{-1/2}(\partial\Omega)^n, \quad \text{for } j = 1, 2.$$

Set  $\pi := \pi_1 - \pi_2 \in L^2(\Omega)$ , and note that

$$(6.5) \quad \nabla \pi = \vec{f}_1 - \vec{f}_2 \in \mathcal{H} \hookrightarrow L^2_1(\Omega)^n.$$

As a consequence,

$$(6.6) \quad \pi \in L^2_1(\Omega).$$

Next, we employ (3.20) and (6.4) in order to write

$$(6.7) \quad \begin{aligned} 0 &= \left\langle \partial_\nu^\lambda(\vec{u}, \pi_1)_{\vec{f}_1} - \partial_\nu^\lambda(\vec{u}, \pi_2)_{\vec{f}_2}, \vec{\psi} \right\rangle \\ &= \left\langle \vec{f}_1, \operatorname{Ex}(\vec{\psi}) \right\rangle + \mathbb{A}_\lambda \left( \nabla \vec{u}, \nabla \operatorname{Ex}(\vec{\psi}) \right) - \left\langle \pi_1, \operatorname{div} \operatorname{Ex}(\vec{\psi}) \right\rangle \\ &\quad - \left\langle \vec{f}_2, \operatorname{Ex}(\vec{\psi}) \right\rangle - \mathbb{A}_\lambda \left( \nabla \vec{u}, \nabla \operatorname{Ex}(\vec{\psi}) \right) + \left\langle \pi_2, \operatorname{div} \operatorname{Ex}(\vec{\psi}) \right\rangle \\ &= \left\langle \vec{f}_1 - \vec{f}_2, \operatorname{Ex}(\vec{\psi}) \right\rangle - \left\langle \pi, \operatorname{div} \operatorname{Ex}(\vec{\psi}) \right\rangle, \end{aligned}$$

for every  $\vec{\psi} \in L^2_{1/2}(\partial\Omega)^n$ . At this stage, we recall (6.5)-(6.6) in order to transform the last expression in (6.7) into

$$(6.8) \quad \langle \nabla \pi, \text{Ex}(\vec{\psi}) \rangle - \langle \pi, \text{div Ex}(\vec{\psi}) \rangle = \langle \text{Tr } \pi, \nu \cdot \vec{\psi} \rangle.$$

In concert with (6.7), this shows that

$$(6.9) \quad \langle (\text{Tr } \pi) \nu, \vec{\psi} \rangle = 0 \quad \text{for every } \vec{\psi} \in L^2_{1/2}(\partial\Omega)^n,$$

from which we may conclude that

$$(6.10) \quad \text{Tr } \pi = 0 \quad \text{in } L^2_{1/2}(\partial\Omega)^n.$$

This, (6.6) and Lemma 6.2 amount to saying that  $\pi \in L^2_1(\Omega)$  is harmonic and satisfies  $\text{Tr } \pi = 0$ . Thus,  $\pi = 0$  in  $\Omega$ , by the uniqueness for the Dirichlet problem. Hence,  $\pi_1 = \pi_2$  in  $\Omega$ , as desired.  $\square$

**Remark.** In particular, Lemma 6.3 implies that there is no ambiguity in defining  $B_\lambda \vec{u}$  as in (6.3).

Recall now the bilinear form (2.33), and consider

$$(6.11) \quad b_\lambda(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}, \quad b_\lambda(\vec{u}, \vec{v}) := \int_\Omega A_\lambda(\overline{\nabla} \vec{u}, \nabla \vec{v}) \, dx.$$

Our goal is to study this sesquilinear form. This requires some prerequisites which we now dispense with. First, the following Korn type estimate has been proved in [23].

**Proposition 6.4.** *Let  $\Omega$  be a Lipschitz domain and assume that  $1 < p < \infty$ . Then there exists a finite constant  $C > 0$  which depends only on  $p$  and the Lipschitz character of  $\Omega$  such that*

$$(6.12) \quad \|\vec{u}\|_{L^p_1(\Omega)^n} \leq C \left\{ \|\nabla \vec{u} + \nabla \vec{u}^\top\|_{L^p(\Omega)^{n^2}} + C \text{diam}(\Omega)^{-1} \|\vec{u}\|_{L^p(\Omega)^n} \right\},$$

uniformly for  $\vec{u} \in L^p_1(\Omega)^n$ .

We shall also need the the following algebraic result from [23].

**Proposition 6.5.** *For every  $\lambda \in (-1, 1]$  there exists  $\kappa_\lambda > 0$  such that for every  $n \times n$ -matrix  $\xi$*

$$(6.13) \quad A_\lambda(\xi, \xi) \geq \kappa_\lambda |\xi|^2 \quad \text{for } |\lambda| < 1 \quad \text{and} \quad A_1(\xi, \xi) \geq \kappa_1 |\xi + \xi^\top|^2.$$

The following well-known result (cf. [4]) is also going to be useful shortly.

**Lemma 6.6.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ , and assume that  $\vec{v} \in [\mathcal{D}(\Omega)']^n$  is a vector-valued distribution which annihilates  $\{\vec{w} \in C^\infty(\Omega)^n : \text{div } \vec{w} = 0 \text{ in } \Omega\}$ . Then there exists a scalar distribution  $q \in \mathcal{D}(\Omega)'$  with the property that  $\vec{v} = \nabla q$  in  $\Omega$ .*

We are now ready to state and prove the main result of this section. Recall the spaces  $\mathcal{V}, \mathcal{H}$  from (2.9), (2.8), along with the form  $b_\lambda(\cdot, \cdot)$  from (6.11).

**Theorem 6.7.** *Let  $\Omega \subset \mathbb{R}^n$  be a Lipschitz domain and assume that  $\lambda \in (-1, 1]$  is fixed. Then the sesquilinear form  $b_\lambda(\cdot, \cdot)$  introduced in (6.11) is symmetric, bounded, non-negative, and closed.*

*Furthermore, the Neumann-Stokes operator  $B_\lambda$ , originally introduced in (6.1)-(6.3), is (in the terminology of § 4) the operator associated with  $b_\lambda(\cdot, \cdot)$ . As a consequence,*

$$(6.14) \quad B_\lambda \text{ is self-adjoint and nonnegative on } \mathcal{H},$$

$$(6.15) \quad D(|B|^{1/2}) = D(B^{1/2}) = \mathcal{V},$$

$$(6.16) \quad D(B) \text{ is dense both in } \mathcal{V} \text{ and in } \mathcal{H}.$$

*Finally,  $\text{Spec}(B_\lambda)$ , the spectrum of the operator (6.1)-(6.3) is a discrete subset of  $[0, \infty)$ .*

*Proof.* Lemma 2.1 ensures that (4.1) holds, hence the formalism from § 4 applies. That the form  $b_\lambda(\cdot, \cdot)$  in (6.11) is nonnegative, symmetric, sesquilinear and continuous is clear from its definition. In addition, this form is coercive, hence closed. Indeed, when  $|\lambda| < 1$  this follows directly from Proposition 6.5, whereas when  $\lambda = 1$  this is a consequence of the second inequality in (6.13) and Proposition 6.4.

We next wish to show the coincidence between the domain  $D(B_\lambda)$  of the Neumann-Stokes operator in (6.2) and the space

$$(6.17) \quad \{\vec{u} \in \mathcal{V} : \text{there exists } \vec{f} \in \mathcal{H} \text{ such that } b_\lambda(\vec{w}, \vec{u}) = (\vec{w}, \vec{f})_{\mathcal{H}} \text{ for all } \vec{w} \in \mathcal{V}\}.$$

In one direction, fix  $\vec{u} \in \mathcal{V}$  such that there exists  $\vec{f} \in \mathcal{H}$  for which

$$(6.18) \quad \int_{\Omega} A_\lambda(\nabla \vec{w}, \nabla \vec{u}) dx = \int_{\Omega} \langle \vec{w}, \vec{f} \rangle dx \quad \text{for every } \vec{w} \in \mathcal{V}.$$

Specializing (6.18) to the case when  $\vec{w} \in C_c^\infty(\Omega)^n$  is divergence-free yields, e.g., on account of (2.34) used with  $\pi = 0$ , that

$$(6.19) \quad \text{the distribution } \vec{f} + \Delta \vec{u} \text{ vanishes on } \{\vec{w} \in C_c^\infty(\Omega)^n : \text{div } \vec{w} = 0 \text{ in } \Omega\}.$$

Then, by virtue of Lemma 6.6, there exists a scalar distribution  $\tilde{\pi}$  in  $\Omega$  such that

$$(6.20) \quad \nabla \tilde{\pi} = \vec{f} + \Delta \vec{u} \in L_{-1}^2(\Omega)^n.$$

Going further, (6.20) and Corollary 3.3 imply that, in fact,

$$(6.21) \quad \tilde{\pi} \in L^2(\Omega) \quad \text{and} \quad \vec{f} = -\Delta \vec{u} + \nabla \tilde{\pi} \text{ in } \Omega.$$

At this point we make the claim that there exists a constant  $c \in \mathbb{R}$  with the property that

$$(6.22) \quad \pi := \tilde{\pi} - c \implies \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}} = 0 \quad \text{in } L_{-1/2}^2(\partial\Omega)^n.$$

To justify this, we first note that (3.23) (used with  $-\vec{f}$  in place of  $\vec{f}$ ) and (6.18) force



$$(6.23) \quad \left\langle \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}}, \operatorname{Tr} \vec{w} \right\rangle = 0 \quad \text{for every } \vec{w} \in \mathcal{V},$$

hence, further,

$$(6.24) \quad \left\langle \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}}, \vec{\varphi} \right\rangle = 0 \quad \text{for every } \vec{\varphi} \in L_{1/2, \nu}^2(\partial\Omega),$$

by Lemma 2.2. To continue, fix some vector field  $\vec{\varphi}_o \in L_{1/2}^2(\partial\Omega)^n$  with the property that  $\int_{\partial\Omega} \nu \cdot \vec{\varphi}_o \, d\sigma = 1$ , and define

$$(6.25) \quad c := \left\langle \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}}, \vec{\varphi}_o \right\rangle.$$

Now, given an arbitrary  $\vec{\varphi} \in L_{1/2}^2(\partial\Omega)^n$ , set  $\lambda := \int_{\partial\Omega} \nu \cdot \vec{\varphi} \, d\sigma$  and compute

$$(6.26) \quad \begin{aligned} \left\langle \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}}, \vec{\varphi} \right\rangle &= \left\langle \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}}, \vec{\varphi} - \lambda \vec{\varphi}_o \right\rangle + \lambda \left\langle \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}}, \vec{\varphi}_o \right\rangle \\ &= 0 + \langle c \nu, \vec{\varphi} \rangle, \end{aligned}$$

by (6.24), (6.25) and the definition of  $\lambda$ . Since  $\vec{\varphi} \in L_{1/2}^2(\partial\Omega)$  is arbitrary, this proves that

$$(6.27) \quad \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}} = c \nu \quad \text{in } L_{-1/2}^2(\partial\Omega)^n.$$

Thus,

$$(6.28) \quad \partial_\nu^\lambda(\vec{u}, \tilde{\pi} - c)_{\vec{f}} = \partial_\nu^\lambda(\vec{u}, \tilde{\pi})_{\vec{f}} - \partial_\nu^\lambda(\vec{0}, c)_{\vec{0}} = c \nu - c \nu = 0 \quad \text{in } L_{-1/2}^2(\partial\Omega)^n,$$

hence (6.22) holds. Note that (6.21) also ensures that  $\pi \in L^2(\Omega)$  and  $\vec{f} = -\Delta \vec{u} + \nabla \pi$  in  $\Omega$ . Together, these conditions prove that the space in (6.17) is contained in  $D(B_\lambda)$  (defined in (6.2)).

Conversely, the inclusion of  $D(B_\lambda)$  into the space in (6.17) is a direct consequence of the definition of the domain of the Neumann-Stokes operator (in (6.2)) and the integration by parts formula (3.23).

Once  $D(B_\lambda)$  has been identified with the space in (6.17), the fact that the Neumann-Stokes operator  $B_\lambda$ , in (6.1)-(6.3) is, in the terminology of § 4, the operator associated with the form  $b_\lambda(\cdot, \cdot)$  follows from (4.34). Finally, the claim made about  $\operatorname{Spec}(B_\lambda)$  is a consequence of the fact that  $B_\lambda$  is nonnegative and has a compact resolvent.  $\square$

## 7. THE STOKES SCALE ADAPTED TO NEUMANN BOUNDARY CONDITIONS

Given a Lipschitz domain  $\Omega \subset \mathbb{R}^n$  and  $1 < p < \infty$ ,  $s \in \mathbb{R}$ , we set

$$(7.1) \quad V^{s,p}(\Omega) := \left\{ \vec{u} \in L_s^p(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega \right\}.$$

The first main result of this section is to show that the above scale is stable under complex interpolation.

**Theorem 7.1.** *For each Lipschitz domain  $\Omega \subset \mathbb{R}^n$ , the family*

$$(7.2) \quad \left\{ V^{s,p}(\Omega) : 1 < p < \infty, s \in \mathbb{R} \right\}$$

*is a complex interpolation scale. In other words, if  $[\cdot, \cdot]_\theta$  stands for the usual complex interpolation bracket, then*

$$(7.3) \quad \left[ V^{s_0, p_0}(\Omega), V^{s_1, p_1}(\Omega) \right]_\theta = V^{s,p}(\Omega)$$

*whenever  $1 < p_i < \infty$ ,  $s_i \in \mathbb{R}$ ,  $i = 0, 1$ ,  $\theta \in [0, 1]$ ,  $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  and  $s := (1-\theta)s_0 + \theta s_1$ .*

Before turning to the proof of Theorem 7.1, we recall a version of an abstract interpolation result from [18].

**Lemma 7.2.** *Let  $X_i, Y_i$ ,  $i = 0, 1$ , be two pairs of Banach spaces such that  $X_0 \cap X_1$  is dense in both  $X_0$  and  $X_1$ , and similarly for  $Y_0, Y_1$ . Let  $D$  be a linear operator such that  $D : X_i \rightarrow Y_i$  boundedly for  $i = 0, 1$ , and consider the following closed subspaces of  $X_i$ ,  $i = 0, 1$ :*

$$(7.4) \quad \text{Ker}(D; X_i) := \{u \in X_i : Du = 0\}, \quad i = 0, 1.$$

*Finally, suppose that there exists a continuous linear mapping  $G : Y_i \rightarrow X_i$  with the property  $D \circ G = I$ , the identity on  $Y_i$  for  $i = 0, 1$ . Then, for each  $0 < \theta < 1$ ,*

$$(7.5) \quad [\text{Ker}(D; X_0), \text{Ker}(D; X_1)]_\theta = \{u \in [X_0, X_1]_\theta : Du = 0\}, \quad \theta \in (0, 1).$$

*Proof of Theorem 7.1.* Denote by  $\Pi$  the harmonic Newtonian potential, i.e., the operator of convolution with the standard fundamental solution for the Laplacian in  $\mathbb{R}^n$ . Recall the universal extension operator  $E_\Omega$  from Theorem 3.1. Without loss of generality, we may assume that  $E_\Omega u$  is supported in a fixed compact neighborhood of  $\Omega$  for every distribution  $u$  in  $\Omega$ . Assuming that this is the case, we set

$$(7.6) \quad \Pi_\Omega := \mathcal{R}_\Omega \circ \Pi \circ E_\Omega,$$

where  $\mathcal{R}_\Omega u := u|_\Omega$  is the operator of restriction to  $\Omega$ . Given that  $\Pi$  is smoothing of order two, it follows that

$$(7.7) \quad \Pi_\Omega : L_s^p(\Omega) \longrightarrow L_{s+2}^p(\Omega), \quad 1 < p < \infty, s \in \mathbb{R},$$

is a well-defined, linear and bounded operator.

Next, fix  $p_0, p_1, p, s_0, s_1, s, \theta$  as in the statement of the theorem. We shall implement Lemma 7.2 in which we take

$$(7.8) \quad X_i := L_{s_i}^{p_i}(\Omega)^n \quad \text{and} \quad Y_i := L_{s_i-1}^{p_i}(\Omega), \quad i = 0, 1,$$

as well as

$$(7.9) \quad D := \text{div} \quad \text{and} \quad G := \nabla \Pi_\Omega.$$

Then since

$$(7.10) \quad D : X_i \longrightarrow Y_i, \quad G : Y_i \longrightarrow X_i, \quad i = 0, 1,$$

are well-defined, linear and bounded, and since  $D \circ G = I$ , the identity, the conclusion in Theorem 7.1 follows from Lemma 7.2.  $\square$

Our next goal is to identify the duals of the spaces in the Stokes scale introduced in (7.1). As a preamble, we prove the following.

**Proposition 7.3.** *Let  $\Omega$  be a Lipschitz domain in  $\mathbb{R}^n$  with outward unit normal  $\nu$  and assume that  $1 < p < \infty$ ,  $-1 + \frac{1}{p} < s < \frac{1}{p}$ . Define the mapping*

$$(7.11) \quad \nu \cdot : V^{s,p}(\Omega) \longrightarrow B_{s-\frac{1}{p}}^{p,p}(\partial\Omega)$$

by setting

$$(7.12) \quad \langle \nu \cdot \vec{u}, \phi \rangle := \langle \vec{u}, \nabla \Phi \rangle$$

for each  $\phi \in \left( B_{s-\frac{1}{p}}^{p,p}(\partial\Omega) \right)^* = B_{-s+\frac{1}{p}}^{p',p'}(\partial\Omega)$ , where  $\Phi \in L_{1-s}^{p'}(\Omega)$  is such that  $\text{Tr } \Phi = \phi$ . Then the above definition is meaningful and the operator (7.11) is bounded in the sense that

$$(7.13) \quad \|\nu \cdot \vec{u}\|_{B_{s-\frac{1}{p}}^{p,p}(\partial\Omega)} \leq C \|\vec{u}\|_{L_s^p(\Omega)^n},$$

for some finite  $C = C(\Omega, s, p) > 0$ . Finally, the range of the operator (7.11)-(7.12) is

$$(7.14) \quad \left\{ f \in B_{s-\frac{1}{p}}^{p,p}(\partial\Omega) : \langle f, 1 \rangle = 0 \right\}.$$

*Proof.* This follows from Proposition 2.7 in [21] and Proposition 2.1 in [20].  $\square$

**Theorem 7.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a Lipschitz domain and fix  $1 < p < \infty$ . Next, for each  $-1 + 1/p < s < 1/p$ , let*

$$(7.15) \quad J_{s,p} : V^{s,p}(\Omega) \hookrightarrow L_s^p(\Omega)^n$$

be the canonical inclusion, and consider its dual

$$(7.16) \quad J_{s,p}^* : L_{-s}^{p'}(\Omega)^n \longrightarrow \left( V^{s,p}(\Omega) \right)^*,$$

where  $1/p + 1/p' = 1$ . Then the mapping (7.16) is onto and its kernel is precisely  $\nabla[L_{1-s,z}^{p'}(\Omega)]$ . In particular,

$$(7.17) \quad J_{s,p}^* : \frac{L_{-s}^{p'}(\Omega)^n}{\nabla[L_{1-s,z}^{p'}(\Omega)]} \longrightarrow \left( V^{s,p}(\Omega) \right)^*$$

is an isomorphism.

*Proof.* Since  $V^{s,p}(\Omega)$  is a closed subspace of  $L_{s,z}^p(\Omega)$ , Hahn-Banach's theorem immediately gives that the mapping (7.16) is onto. That (7.17) is an isomorphism will then follow as soon as we show that  $\text{Ker } J_{s,p}^*$ , the null-space of the application (7.16), coincides with  $\nabla[L_{1-s,z}^{p'}(\Omega)]$ . In one direction, if  $\vec{u} \in L_{-s}^{p'}(\Omega)^n = \left(L_s^p(\Omega)^n\right)^*$  is such that  $J_{s,p}^*(\vec{u}) = 0$ , then  $\langle \vec{u}, \vec{v} \rangle = 0$  for each  $\vec{v} \in V^{s,p}(\Omega)$ . Choosing  $\vec{v} \in C_c^\infty(\Omega)^n$  such that  $\text{div } \vec{v} = 0$  in  $\Omega$  shows, on account of Lemma 6.6, that there exists a distribution  $w$  in  $\Omega$  such that  $\nabla w = \vec{u}$ . Proposition 3.2 then ensures that  $w \in L_{1-s}^{p'}(\Omega)$ , so that  $\vec{u} = \nabla w \in \nabla[L_{1-s}^{p'}(\Omega)]$ . There remains to show that, after subtracting a suitable constant from  $w$ , this function can be made to have trace zero and, hence, belong to  $L_{1-s,z}^{p'}(\Omega)$ . To this end, note that for each  $\vec{v} \in V^{s,p}(\Omega)$  we have

$$(7.18) \quad 0 = \langle \vec{u}, \vec{v} \rangle = \langle \nabla, \vec{v} \rangle = \langle \text{Tr } w, \nu \cdot \vec{v} \rangle.$$

Then the last claim in Proposition 7.3 shows that  $\text{Tr } w$  is a constant, as wanted.

Conversely, if  $\vec{u} = \nabla \Phi \in L_{-s}^{p'}(\Omega, \mathbb{R}^n)$  for some  $\Phi \in L_{1-s,z}^{p'}(\Omega)$  then Proposition 7.3 allows us to write

$$(7.19) \quad \langle J_{s,p}^*(\vec{u}), \vec{v} \rangle = \langle \nabla \Phi, \vec{v} \rangle = \langle \text{Tr } \Phi, \nu \cdot \vec{v} \rangle = 0,$$

for every  $\vec{v} \in V^{s,p}(\Omega)$ . Thus,  $J_{s,p}^*(\vec{u}) = 0$ , finishing the proof of the theorem.  $\square$

**Theorem 7.5.** *For each Lipschitz domain  $\Omega \subset \mathbb{R}^n$  there exists  $\varepsilon = \varepsilon(\Omega) \in (0, 1]$  with the following significance. Assume that  $1 < p < \infty$ ,  $-1 + 1/p < s < 1/p$  and that the pair  $(s, 1/p)$  satisfies either of the following three conditions:*

$$(7.20) \quad \begin{aligned} (I) : & \quad 0 < \frac{1}{p} < \frac{1-\varepsilon}{2} \quad \text{and} \quad -1 + \frac{1}{p} < s < \frac{3}{p} - 1 + \varepsilon; \\ (II) : & \quad \frac{1-\varepsilon}{2} \leq \frac{1}{p} \leq \frac{1+\varepsilon}{2} \quad \text{and} \quad -1 + \frac{1}{p} < s < \frac{1}{p}; \\ (III) : & \quad \frac{1+\varepsilon}{2} < \frac{1}{p} < 1 \quad \text{and} \quad -2 + \frac{3}{p} - \varepsilon < s < \frac{1}{p}. \end{aligned}$$

Then

$$(7.21) \quad L_s^p(\Omega)^n = V^{s,p}(\Omega) \oplus \nabla[L_{s+1,z}^p(\Omega)],$$

where the direct sum is topological (in fact, orthogonal when  $s = 0$  and  $p = 2$ ). Furthermore, if

$$(7.22) \quad \mathbb{P} : L_s^p(\Omega)^n \longrightarrow V^{s,p}(\Omega)$$

denotes the projection onto the first summand in the decomposition (7.21), then its kernel is  $\nabla[L_{s+1,z}^p(\Omega)]$ . In particular,

$$(7.23) \quad \mathbb{P} : \frac{L_s^p(\Omega)^n}{\nabla[L_{s+1,z}^p(\Omega)]} \longrightarrow V^{s,p}(\Omega)$$

is an isomorphism. Also, the adjoint of the operator

$$(7.24) \quad \mathbb{P}_{p,s} : L_s^p(\Omega)^n \xrightarrow{\mathbb{P}} V^{s,p}(\Omega) \xrightarrow{J^{s,p}} L_s^p(\Omega)^n$$

is the operator  $\mathbb{P}_{p',-s}$ , and

$$(7.25) \quad \left( V^{s,p}(\Omega) \right)^* = V^{-s,p'}(\Omega).$$

*Proof.* The decomposition (7.21) corresponding to the case when  $s = 0$  has been established in [8] via an approach which reduces matters to the well-posedness of the inhomogeneous Dirichlet problem for the Laplacian in the Lipschitz domain  $\Omega$ . The more general case considered here can be proved in an analogous fashion. With (7.21) in hand, the claims about the projection (7.22) are straightforward.

Consider next the identification in (7.25). If  $\vec{u} \in V^{-s,p'}(\Omega)$  define  $\Lambda_{\vec{u}} \in \left( V^{s,p}(\Omega) \right)^*$  by setting

$$(7.26) \quad \Lambda_{\vec{u}}(\vec{v}) := {}_{L_s^p(\Omega)^n} \langle \vec{v}, \vec{u} \rangle_{L_{-s}^{p'}(\Omega)^n}, \quad \forall \vec{v} \in V^{s,p}(\Omega).$$

Then the mapping

$$(7.27) \quad \Phi : V^{-s,p'}(\Omega) \longrightarrow \left( V^{s,p}(\Omega) \right)^*, \quad \Phi(\vec{u}) := \Lambda_{\vec{u}},$$

is well-defined, linear and bounded. Our goal is to show that this is an isomorphism. To prove that  $\Phi$  is onto, fix  $\Lambda \in \left( V^{s,p}(\Omega) \right)^*$ . Recall the operator  $\mathbb{P}$  from (7.22) and note that  $\Lambda \circ \mathbb{P} \in \left( L_s^p(\Omega) \right)^* = L_{s'}^{p'}(\Omega)$ . That is, there exists  $\vec{w} \in L_{s'}^{p'}(\Omega)$  such that

$$(7.28) \quad (\Lambda \circ \mathbb{P})\vec{u} = {}_{L_s^p(\Omega)^n} \langle \vec{w}, \vec{u} \rangle_{L_{-s}^{p'}(\Omega)^n}, \quad \forall \vec{u} \in V^{s,p}(\Omega).$$

Then  $\Lambda_{\mathbb{P}\vec{w}} := \Phi(\mathbb{P}\vec{w})$  satisfies

$$(7.29) \quad \begin{aligned} \Lambda_{\mathbb{P}\vec{w}}(\vec{v}) &= \langle \vec{v}, \mathbb{P}\vec{w} \rangle = \overline{\langle \vec{w}, \mathbb{P}\vec{v} \rangle} = \overline{\langle \vec{w}, \vec{v} \rangle} \\ &= \langle \vec{v}, \vec{w} \rangle = (\Lambda \circ \mathbb{P})\vec{v} = \Lambda(\vec{v}), \quad \forall \vec{v} \in V^{s,p}(\Omega). \end{aligned}$$

Hence  $\Lambda = \Lambda_{\mathbb{P}\vec{w}}$ , proving that  $\Phi$  is onto. To see that  $\Phi$  is also one-to-one, we note that if  $\vec{u} \in V^{-s,p'}(\Omega)$  is such that  $\Lambda_{\vec{u}} = 0$ , then

$$(7.30) \quad \begin{aligned} \langle \vec{u}, \vec{v} \rangle = 0 \quad \forall \vec{v} \in V^{s,p}(\Omega) &\implies \langle \vec{u}, \mathbb{P}\vec{w} \rangle = 0 \quad \forall \vec{w} \in L_s^p(\Omega) \\ &\implies \langle \mathbb{P}\vec{u}, \vec{w} \rangle = 0 \quad \forall \vec{w} \in L_s^p(\Omega) \\ &\implies \langle \vec{u}, \vec{w} \rangle = 0 \quad \forall \vec{w} \in L_s^p(\Omega) \\ &\implies \vec{u} = 0. \end{aligned}$$

This shows that  $\Phi$  in (7.27) is an isomorphism, thus finishing the proof of (7.25). The proof of the theorem is therefore completed.  $\square$

## 8. THE POISSON PROBLEM FOR THE STOKES OPERATOR WITH NEUMANN CONDITIONS

For a given Lipschitz domain  $\Omega$  in  $\mathbb{R}^n$ ,  $n \geq 2$ , the range of indices for which the Poisson problem in  $\Omega$  for the Stokes operator equipped with Neumann boundary conditions is well-posed on Besov and Triebel-Lizorkin spaces depends on the dimension  $n$  of the ambient space and the Lipschitz character of  $\Omega$ . The latter is manifested by a parameter  $\varepsilon \in (0, 1]$  which can be thought of as measuring the degree of roughness of  $\Omega$  (thus, the larger  $\varepsilon$  the milder the Lipschitz nature of  $\Omega$ , and the smaller  $\varepsilon$ , the more acute Lipschitz nature of  $\Omega$ ). To best describe these regions, for each  $n \geq 2$  and  $\varepsilon > 0$  we let  $\mathcal{R}_{n,\varepsilon}$  denote the following sets. For  $n = 2$ ,  $\mathcal{R}_{2,\varepsilon}$  is the collection of all pairs of numbers  $s, p$  with the property that either one of the following two conditions below is satisfied:

$$(8.1) \quad \begin{aligned} (I_2) : \quad & 0 \leq \frac{1}{p} < s + \frac{1+\varepsilon}{2} \quad \text{and} \quad 0 < s \leq \frac{1+\varepsilon}{2}, \\ (II_2) : \quad & -\frac{1+\varepsilon}{2} < \frac{1}{p} - s < \frac{1+\varepsilon}{2} \quad \text{and} \quad \frac{1+\varepsilon}{2} < s < 1. \end{aligned}$$

Corresponding to  $n = 3$ ,  $\mathcal{R}_{3,\varepsilon}$  is the collection of all pairs  $s, p$  with the property that either of the following two conditions holds:

$$(8.2) \quad \begin{aligned} (I_3) : \quad & 0 \leq \frac{1}{p} < \frac{s}{2} + \frac{1+\varepsilon}{2} \quad \text{and} \quad 0 < s < \varepsilon, \\ (II_3) : \quad & -\frac{\varepsilon}{2} < \frac{1}{p} - \frac{s}{2} < \frac{1+\varepsilon}{2} \quad \text{and} \quad \varepsilon \leq s < 1. \end{aligned}$$

Finally, corresponding to  $n \geq 4$ , we let  $\mathcal{R}_{n,\varepsilon}$  denote the collection of all pairs  $s, p$  with the property that

$$(8.3) \quad (I_n) : \quad \frac{n-3}{2(n-1)} - \varepsilon < \frac{1}{p} - \frac{s}{n-1} < \frac{1}{2} + \varepsilon \quad \text{and} \quad 0 < s < 1, \quad 1 < p < \infty.$$

The following well-posedness result has been recently established in [23].

**Theorem 8.1.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with connected complement, and fix  $\frac{n-1}{n} < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $(n-1)\left(\frac{1}{p} - 1\right)_+ < s < 1$ . Also, assume that  $\lambda \in (-1, 1]$  and  $\mu \in \mathbb{C} \setminus \text{Spec}(B_\lambda)$ . Then there exists  $\varepsilon = \varepsilon(\Omega) \in (0, 1]$  such that the Poisson problem for the Stokes system with Neumann boundary condition*

$$(8.4) \quad \begin{aligned} \mu \vec{u} - \Delta \vec{u} + \nabla \pi &= \vec{f} \Big|_{\Omega}, \quad \vec{f} \in B_{s+\frac{1}{p}-2,0}^{p,q}(\Omega), \quad \text{div } \vec{u} = 0 \text{ in } \Omega, \\ \vec{u} \in B_{s+\frac{1}{p}}^{p,q}(\Omega), \quad \pi &\in B_{s+\frac{1}{p}-1}^{p,q}(\Omega), \quad \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}-\mu \vec{u}} = 0 \text{ in } B_{s-1}^{p,q}(\partial\Omega), \end{aligned}$$

has a unique solution if the pair  $s, p$  belongs to the region  $\mathcal{R}_{n,\varepsilon}$ , described in (8.1)-(8.3). In addition, the solution satisfies the estimate

$$(8.5) \quad \|\vec{u}\|_{B_{s+\frac{1}{p}}^{p,q}(\Omega)} + \|\pi\|_{B_{s+\frac{1}{p}-1}^{p,q}(\Omega)} \leq C \|\vec{f}\|_{B_{s+\frac{1}{p}-2,0}^{p,q}(\Omega)},$$

for some finite constant  $C = C(\Omega, n, p, s, \lambda, \mu) > 0$ .

Moreover, an analogous well-posedness result holds for the problem

$$(8.6) \quad \begin{aligned} \mu \vec{u} - \Delta \vec{u} + \nabla \pi &= \vec{f} \Big|_{\Omega}, \quad \vec{f} \in F_{s+\frac{1}{p}-2,0}^{p,q}(\Omega), \quad \operatorname{div} \vec{u} = 0 \text{ in } \Omega, \\ \vec{u} \in F_{s+\frac{1}{p}}^{p,q}(\Omega), \quad \pi &\in F_{s+\frac{1}{p}-1}^{p,q}(\Omega), \quad \partial_{\nu}^{\lambda}(\vec{u}, \pi)_{\vec{f}-\mu \vec{u}} = 0 \text{ in } B_{s-1}^{p,p}(\partial\Omega), \end{aligned}$$

assuming that  $p, q < \infty$ .

Strictly speaking, the above theorem has been proved in [23] when  $\mu = 0$  (in which case the data must satisfy certain necessary compatibility conditions, and uniqueness is valid up a finite dimensional space). The method of proof in [23] is constructive as it relies on integral representation formulas (involving hydrostatic potential operators). As such, this approach can be easily adapted to the slightly more general case above, since the difference between the fundamental solutions for the original Stokes system  $\{-\Delta \vec{u} + \nabla \pi = 0, \operatorname{div} \vec{u} = 0\}$  and the lower-order perturbation  $\{(\mu - \Delta) \vec{u} + \nabla \pi = 0, \operatorname{div} \vec{u} = 0\}$  is only weakly singular. We leave the straightforward details to the interested reader.

## 9. DOMAINS OF FRACTIONAL POWERS OF THE NEUMANN STOKES OPERATOR: I

Here we study the global regularity, measured on the Sobolev scale, of vector fields in the domains of fractional powers of the Neumann Stokes operator. Our first result in this regard reads as follows:

**Theorem 9.1.** *Let  $\Omega$  be a Lipschitz domain in  $\mathbb{R}^n$  and fix  $\lambda \in (-1, 1]$ . Then the domain of the fractional power of the Neumann Stokes operator  $B_{\lambda}$  introduced in (6.2)-(6.3) satisfies*

$$(9.1) \quad D(B_{\lambda}^{\frac{s}{2}}) = \left\{ \vec{u} \in L_s^2(\Omega)^n : \operatorname{div} \vec{u} = 0 \right\} \text{ if } 0 \leq s \leq 1,$$

and

$$(9.2) \quad \vec{u} \in D(B_{\lambda}^{\frac{s}{2}}) \iff \begin{cases} \vec{u} \in \mathcal{V} \text{ and there exists } \pi \in L^2(\Omega) \text{ such that} \\ \vec{f} := (1 - \Delta) \vec{u} + \nabla \pi \in L_{s-2}^2(\Omega)^n \text{ and } \partial_{\nu}^{\lambda}(\vec{u}, \pi)_{\vec{f}-\vec{u}} = 0, \end{cases}$$

granted that  $s \in (3/2, 2]$ .

*Proof.* Consider the families of spaces  $\{V^{s,2}(\Omega) : s \in \mathbb{R}\}$  and  $\{D(B_{\lambda}^{\frac{s}{2}}) : 0 \leq s \leq 2\}$ . From Theorem 7.1 and (5.11) we know that both are complex interpolation scales, and

$$(9.3) \quad D(B_{\lambda}^0) = \mathcal{H} = V^{0,2}(\Omega), \quad D(B_{\lambda}^{\frac{1}{2}}) = \mathcal{V} = V^{1,2}(\Omega).$$

Thus, by complex interpolation,

$$(9.4) \quad D(B_{\lambda}^{\frac{s}{2}}) = V^{s,2}(\Omega), \quad 0 \leq s \leq 1,$$

which gives the description of  $D(B_{\lambda}^{\frac{s}{2}})$  in (9.1).

To study larger values of  $s$ , recall the form  $b_{\lambda}(\cdot, \cdot)$  and the operator  $\tilde{B}_{\lambda}$  induced by it. From (5.20)-(9.4) we obtain

$$(9.5) \quad D(B_\lambda^{\frac{s}{2}}) = (\tilde{I} + \tilde{B}_\lambda)^{-1} \left( V^{2-s,2}(\Omega) \right)^*, \quad 1 \leq s \leq 2.$$

Thus, by (7.25),

$$(9.6) \quad \vec{u} \in D(B_\lambda^{\frac{s}{2}}) \iff \vec{u} \in \mathcal{V} \text{ and } (\tilde{I} + \tilde{B}_\lambda)\vec{u} \in V^{s-2,2}(\Omega), \quad \text{if } \frac{3}{2} < s \leq 2.$$

Consequently, if  $s \in (3/2, 2]$ , then by taking into account the very definition of  $\tilde{B}_\lambda$  we arrive at the conclusion that

$$(9.7) \quad \vec{u} \in D(B_\lambda^{\frac{s}{2}}) \iff \begin{cases} \vec{u} \in \mathcal{V} \text{ and } \exists \vec{f} \in L_{s-2}^2(\Omega)^n \text{ such that} \\ \langle \vec{f}, \vec{v} \rangle = \int_\Omega \vec{u} \cdot \vec{v} dx + \int_\Omega A_\lambda(\nabla \vec{u}, \nabla \vec{v}) dx, \quad \forall \vec{v} \in \mathcal{V}. \end{cases}$$

Much as before, by relying on Lemma 6.6, Corollary 3.3 and Proposition 3.5, it follows from (9.7) that (9.2) holds.  $\square$

It is possible to further extend the scope of the above analysis. In order to facilitate the subsequent discussion, for each  $\varepsilon \in (0, 1]$ ,  $s \in [\frac{3}{2}, 2]$  and  $n \geq 2$ , define the two dimensional region

$$(9.8) \quad R_{n,s,\varepsilon} := \begin{cases} (\theta, \frac{1}{p}) : 0 < \frac{1}{p} < \theta < 1 + \frac{1}{p} < 2, \quad \theta \leq s, \quad \text{and} \\ \frac{1}{2} + \varepsilon > \frac{1}{p} - \frac{\theta}{n} \geq \frac{1}{2} - \frac{s}{n} \quad \text{if} \quad \frac{3}{2} \leq s < \frac{n}{n-1} + \varepsilon n, \\ \frac{1}{2} + \varepsilon > \frac{1}{p} - \frac{\theta}{n} > -\frac{\varepsilon}{n} \quad \text{if} \quad \frac{n}{n-1} + \varepsilon n < s \leq 2. \end{cases}$$

The figures below depict the region  $R_{n,s,\varepsilon}$  in the case when  $\frac{3}{2} \leq s < \frac{n}{n-1} + \varepsilon n$ ,

and when  $\frac{n}{n-1} + \varepsilon n < s \leq 2$ , respectively:

**Theorem 9.2.** *For every Lipschitz domain  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , there exists  $\varepsilon = \varepsilon(\partial\Omega) > 0$  with the property that for every  $s \in (3/2, 2]$  and  $\lambda \in (-1, 1]$  the following implication holds:*

$$(9.9) \quad (\theta, 1/p) \in R_{n,s,\varepsilon} \implies D(B_\lambda^{s/2}) \subset L_\theta^p(\Omega)^n.$$

*Proof.* The strategy is to combine the characterization (9.2) with the well-posedness result for the Poisson problem for the Stokes system equipped with Neumann boundary conditions. In concert, these two results show that  $D(B_\lambda^{\alpha/2}) \subset L_\theta^p(\Omega)^n$  provided

$$(9.10) \quad \begin{aligned} &\exists s, p \text{ belonging to the region } \mathcal{R}_{n,\varepsilon} \text{ such that} \\ &\theta = s + 1/p \quad \text{and} \quad L_{\alpha-2}^2(\Omega) \hookrightarrow L_{\theta-2}^p(\Omega). \end{aligned}$$

Now, elementary algebra shows that, given  $\alpha \in (3/2, 2]$ , the condition (9.10) holds if and only if  $(\theta, 1/p) \in R_{n,\alpha,\varepsilon}$ . Clearly, this proves (9.9), after re-adjusting notation.  $\square$



**Corollary 9.3.** *For a Lipschitz domain  $\Omega$  in  $\mathbb{R}^n$  one has*

$$(9.11) \quad D(B_\lambda^\alpha) \subset \bigcup_{p > \frac{2n}{n-1}} L_1^p(\Omega)^n \quad \text{if } \alpha > \frac{3}{4}.$$

Also, when  $n = 3$ ,

$$(9.12) \quad D(B_\lambda^\alpha) \subset C^{2\alpha-3/2}(\bar{\Omega})^3 \quad \text{if } \frac{3}{4} < \alpha < \frac{3}{4} + \varepsilon,$$

and when  $n = 2$ ,

$$(9.13) \quad D(B_\lambda^\alpha) \subset C^{2\alpha-1}(\bar{\Omega})^2 \quad \text{if } \frac{3}{4} < \alpha < \frac{3}{4} + \varepsilon,$$

for some small  $\varepsilon = \varepsilon(\Omega) > 0$ .

*Proof.* These are all immediate consequences of Theorem 9.2 and classical embeddings.  $\square$

## 10. DOMAINS OF FRACTIONAL POWERS OF THE NEUMANN STOKES OPERATOR: II

The aim of this section is to augment the results in Theorem 9.1 by including a description of  $D(B_\lambda^{s/2})$  in the case when  $s \in (1, 3/2]$ . See Theorem 10.4 below. We begin by revisiting the Neumann-Leray projection (7.22), with the goal of further extending the range of action of this operator.

**Lemma 10.1.** *Assume that  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^n$  and that  $s \in \mathbb{R}$ ,  $p, p' \in (1, \infty)$ ,  $1/p + 1/p' = 1$ . Then the operator*

$$(10.1) \quad \widehat{\mathbb{P}}_{s,p} : L_{-s,0}^{p'}(\Omega)^n = (L_s^p(\Omega)^n)^* \longrightarrow (V^{s,p}(\Omega))^*$$

defined by the requirement that

$$(10.2) \quad {}_{V^{s,p}(\Omega)} \langle \vec{v}, \widehat{\mathbb{P}}_{s,p} \vec{u} \rangle_{(V^{s,p}(\Omega))^*} = {}_{L_s^p(\Omega)^n} \langle \vec{v}, \vec{u} \rangle_{(L_s^p(\Omega)^n)^*} \quad \forall \vec{v} \in V^{s,p}(\Omega),$$

is well-defined, linear, bounded and onto. Furthermore, any two such operators act coherently, i.e.,  $\widehat{\mathbb{P}}_{s_1,p_1} = \widehat{\mathbb{P}}_{s_2,p_2}$  on  $L_{-s_1,0}^{p_1}(\Omega)^n \cap L_{-s_2,0}^{p_2}(\Omega)^n$  for any numbers  $s_1, s_2 \in \mathbb{R}$  and  $p_1, p_2 \in (1, \infty)$ . Next, if corresponding to  $s = 1$  and  $p = 2$  one considers

$$(10.3) \quad \begin{aligned} \widehat{\mathbb{P}}_{1,2} : L_{-1,0}^2(\Omega)^n &= (L_1^2(\Omega)^n)^* \longrightarrow \mathcal{V}^*, \\ \mathcal{V} \langle \vec{v}, \widehat{\mathbb{P}}_{1,2} \vec{u} \rangle_{\mathcal{V}^*} &= {}_{L_1^2(\Omega)^n} \langle \vec{v}, \vec{u} \rangle_{(L_1^2(\Omega)^n)^*} \quad \forall \vec{v} \in \mathcal{V}, \end{aligned}$$

then the diagram

$$(10.4) \quad \begin{array}{ccc} L_{-1,0}^2(\Omega)^n & \xrightarrow{\widehat{\mathbb{P}}_{1,2}} & \mathcal{V}^* \\ \uparrow & & \uparrow \\ L^2(\Omega)^n & \xrightarrow{\mathbb{P}} & \mathcal{H} \end{array}$$

in which the vertical arrows are natural inclusions, is commutative. Consequently, the Neumann-Leray projection (7.22) extends as in (10.3).

*Proof.* That (10.1)-(10.2) is well-defined and bounded is clear from the continuity of the inclusion  $V^{s,p}(\Omega) \hookrightarrow L_s^p(\Omega)^n$ . Using the fact that  $V^{s,p}(\Omega)$  is a closed subspace of  $L_s^p(\Omega)^n$ , Hahn-Banach theorem, and (3.6), it is straightforward to show that the operator (10.1) is onto. It is also clear from (10.2) that this family of operators act in a mutually compatible fashion.

To show that the diagram (10.4) is commutative, pick  $\vec{u} \in L^2(\Omega)^n$  and use (7.21) (with  $s = 0$  and  $p = 2$ ) in order to decompose it as  $\mathbb{P}\vec{u} + \nabla\pi$  for some  $\pi \in L_{1,z}^2(\Omega)$ . Then, since

$$(10.5) \quad (\nabla\pi, \vec{v})_{L^2(\Omega)^n} = 0 \quad \forall \vec{v} \in \mathcal{V},$$

for every  $\vec{v} \in \mathcal{V}$  we have

$$(10.6) \quad \begin{aligned} \nu \langle \vec{v}, \widehat{\mathbb{P}}_{1,2}\vec{u} \rangle_{\mathcal{V}^*} &= L_1^2(\Omega)^n \langle \vec{v}, \vec{u} \rangle_{(L_1^2(\Omega)^n)^*} = (\vec{v}, \vec{u})_{L^2(\Omega)^n} \\ &= \langle \vec{v}, \mathbb{P}\vec{u} \rangle_{L^2(\Omega)^n} + (\vec{v}, \nabla\pi)_{L^2(\Omega)^n} = \nu \langle \vec{v}, \mathbb{P}\vec{u} \rangle_{\mathcal{V}^*}. \end{aligned}$$

This shows that  $\widehat{\mathbb{P}}_{1,2}\vec{u} = \mathbb{P}\vec{u}$  in  $\mathcal{V}^*$ , as desired.  $\square$

Let  $\Omega \subset \mathbb{R}^n$  be a Lipschitz domain and assume that  $\lambda \in (-1, 1]$  has been fixed. Recall the operator  $\widetilde{B}_\lambda$  induced by the sesquilinear form  $b_\lambda(\cdot, \cdot)$ , i.e.,

$$(10.7) \quad \widetilde{B}_\lambda : \mathcal{V} \longrightarrow \mathcal{V}^*, \quad \widetilde{B}_\lambda \vec{u} := b_\lambda(\cdot, \vec{u}) \in \mathcal{V}^*, \quad \vec{u} \in \mathcal{V}.$$

Next, fix  $\vec{u} \in V$ , so that  $\widetilde{B}_\lambda \vec{u} : \mathcal{V} \rightarrow \mathbb{C}$  is a linear, bounded functional. Since  $\mathcal{V}$  is a closed subspace of  $L_1^2(\Omega)^n$ , the Hahn-Banach theorem ensures the existence of some linear, bounded functional  $\vec{f} : L_1^2(\Omega)^n \rightarrow \mathbb{C}$  with the property that  $\vec{f}|_{\mathcal{V}} = (\widetilde{I} + \widetilde{B}_\lambda)\vec{u}$ . Thus,  $\vec{f} \in (L_1^2(\Omega)^n)^* = L_{1,0}^2(\Omega)^n$  satisfies

$$(10.8) \quad \begin{aligned} L_1^2(\Omega)^n \langle \vec{v}, \vec{f} \rangle_{(L_1^2(\Omega)^n)^*} &= \nu \langle \vec{v}, (\widetilde{I} + \widetilde{B}_\lambda)\vec{u} \rangle_{\mathcal{V}^*} \\ &= \int_{\Omega} \vec{u} \cdot \vec{v} \, dx + \int_{\Omega} A_\lambda(\overline{\nabla\vec{u}}, \nabla\vec{v}) \, dx, \quad \vec{v} \in \mathcal{V} \hookrightarrow L_1^2(\Omega)^n. \end{aligned}$$

Specializing this to the case when  $\vec{v}$  belongs to  $\{\vec{v} \in C_c^\infty(\Omega)^n : \operatorname{div} \vec{v} = 0 \text{ in } \Omega\}$  shows that the distribution  $\vec{f}|_{\Omega} - (1 - \Delta)\vec{u} \in L_{-1}^2(\Omega)^n$  annihilates this space. Thus, by Lemma 6.6, there exists a distribution  $\pi$  in  $\Omega$  such that

$$(10.9) \quad \nabla\pi = \vec{f}|_{\Omega} - (1 - \Delta)\vec{u} \in L_{-1}^2(\Omega)^n.$$

In particular,  $\pi \in L^2(\Omega)$  by Corollary 3.3. Returning with this information back in (10.8) and invoking (3.23) then shows that, after an eventual re-normalization of  $\pi$  (done by subtracting a suitable constant, similar in spirit to (6.22)), matters can be arranged so that

$$(10.10) \quad \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}-\vec{u}} = 0 \quad \text{in } L_{-1/2}^2(\partial\Omega)^n.$$

The stage is now set for proving the following result.

**Proposition 10.2.** *Suppose that  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain and assume that  $\lambda \in (-1, 1]$ . Then for every  $\vec{u} \in \mathcal{V}$  there exist*

$$(10.11) \quad \pi \in L^2(\Omega) \quad \text{and} \quad \vec{f} \in L^2_{-1,0}(\Omega)^n$$

such that

$$(10.12) \quad (1 - \Delta)\vec{u} + \nabla\pi = \vec{f}|_{\Omega} \quad \text{in } L^2_{-1}(\Omega)^n,$$

$$(10.13) \quad \partial_{\nu}^{\lambda}(\vec{u}, \pi)_{\vec{f}-\vec{u}} = 0 \quad \text{in } L^2_{-1/2}(\partial\Omega)^n,$$

$$(10.14) \quad \text{and } (\tilde{I} + \tilde{B}_{\lambda})\vec{u} = \widehat{\mathbb{P}}_{1,2}\vec{f} \quad \text{in } \mathcal{V}^*.$$

Furthermore, if  $\vec{g} \in L^2_{-1,0}(\Omega)^n$  is such that  $\widehat{\mathbb{P}}_{1,2}\vec{g} = \widehat{\mathbb{P}}_{1,2}\vec{f}$ , then there exists  $q \in L^2(\Omega)$  with the property that

$$(10.15) \quad (1 - \Delta)\vec{u} + \nabla(\pi - q) = \vec{g}|_{\Omega} \quad \text{in } L^2_{-1}(\Omega)^n,$$

$$(10.16) \quad \partial_{\nu}^{\lambda}(\vec{u}, \pi - q)_{\vec{g}-\vec{u}} = 0 \quad \text{in } L^2_{-1/2}(\partial\Omega)^n.$$

*Proof.* The existence of  $\pi, \vec{f}$  as in (10.11) and for which (10.12)-(10.13) are satisfied is clear from the discussion preceding the statement of the proposition. Hence, there remains to prove (10.14). This, however, is a direct consequence of Lemma 10.1 and the first equality in (10.8).

There remains to take care of the claim in the second part of the statement. To this end, we first note that  $\widehat{\mathbb{P}}_{1,2}(\vec{f} - \vec{g}) = 0$  entails

$$(10.17) \quad L^2_1(\Omega)^n \langle \vec{v}, \vec{f} - \vec{g} \rangle_{(L^2_1(\Omega)^n)^*} = 0, \quad \forall \vec{v} \in \mathcal{V}.$$

Thus, via a familiar (by now) argument based on Lemma 6.6 and Corollary 3.3, we see that there exists some scalar function  $\hat{q} \in L^2(\Omega)$  with the property that  $(\vec{f} - \vec{g})|_{\Omega} = \nabla\hat{q}$  in  $L^2_{-1}(\Omega)$ . In turn, this and (10.12) yield

$$(10.18) \quad (1 - \Delta)\vec{u} + \nabla(\pi - \hat{q}) = \vec{g}|_{\Omega} \quad \text{in } L^2_{-1}(\Omega)^n.$$

Going further, formula (3.23) gives that for every  $\vec{w} \in \mathcal{V}$

$$(10.19) \quad \begin{aligned} \left\langle \text{Tr } \vec{w}, \partial_{\nu}^{\lambda}(\vec{u}, \pi - \hat{q})_{\vec{g}-\vec{u}} \right\rangle &= \int_{\Omega} \overline{\vec{w}} \cdot \vec{u} \, dx + \mathbb{A}_{\lambda}(\overline{\nabla\vec{w}}, \nabla\vec{u}) \\ &\quad - L^2_1(\Omega)^n \langle \vec{w}, \vec{g} \rangle_{(L^2_1(\Omega)^n)^*}. \end{aligned}$$

On the other hand, for every  $\vec{w} \in \mathcal{V}$  we have

$$\begin{aligned}
L_1^2(\Omega)^n \langle \vec{w}, \vec{g} \rangle_{(L_1^2(\Omega)^n)^*} &= \nu \langle \vec{w}, \widehat{\mathbb{P}}_{1,2} \vec{g} \rangle_{\mathcal{V}^*} = \nu \langle \vec{w}, \widehat{\mathbb{P}}_{1,2} \vec{f} \rangle_{\mathcal{V}^*} \\
&= L_1^2(\Omega)^n \langle \vec{w}, \vec{f} \rangle_{(L_1^2(\Omega)^n)^*} \\
(10.20) \qquad \qquad \qquad &= \int_{\Omega} \vec{w} \cdot \vec{u} \, dx + \mathbb{A}_\lambda(\overline{\nabla} \vec{w}, \nabla \vec{u}),
\end{aligned}$$

by hypotheses, (10.3), (3.23) and (10.13). Together, this and (10.19) then prove that

$$(10.21) \qquad \left\langle \partial_\nu^\lambda(\vec{u}, \pi - \hat{q})_{\vec{g}-\vec{u}}, \text{Tr } \vec{w} \right\rangle = 0, \quad \forall \vec{w} \in \mathcal{V}.$$

With this in hand, and by proceeding as in (6.23)-(6.28), we may then conclude that there exists a constant  $c \in \mathbb{R}$  with the property that if  $q := \hat{q} - c$  then (10.15)-(10.16) hold.  $\square$

Once again, suppose that  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain and that  $\lambda \in (-1, 1]$ . Also, fix  $p \in (1, \infty)$  and assume that  $1/p < s < 1 + 1/p$ ,  $1 < p' < \infty$ ,  $1/p + 1/p' = 1$ . Then the operator  $\widetilde{B}_\lambda$  from (10.7) extends to a bounded mapping

$$\begin{aligned}
(10.22) \qquad \widetilde{B}_\lambda : V^{s,p}(\Omega) &\longrightarrow \left( V^{2-s,p'}(\Omega) \right)^*, \\
\widetilde{B}_\lambda \vec{u} := \mathbb{A}_\lambda(\cdot, \vec{u}) &\in \left( V^{2-s,p'}(\Omega) \right)^*, \quad \vec{u} \in V^{s,p}(\Omega).
\end{aligned}$$

A similar line of reasoning as in the proof of Proposition 10.2 (the only significant difference is that Proposition 3.2 is used in place of Corollary 3.3) then yields the following.

**Proposition 10.3.** *Retain the above notation and conventions. Also, assume that  $\mu \in \mathbb{R}$ . Then for every  $\vec{u} \in V^{s,p}(\Omega)$  there exist*

$$(10.23) \qquad \pi \in L_{s-1}^p(\Omega) \quad \text{and} \quad \vec{f} \in L_{s+1/p-2,0}^p(\Omega)^n$$

such that

$$(10.24) \qquad (\mu - \Delta) \vec{u} + \nabla \pi = \vec{f} \Big|_{\Omega} \quad \text{in } L_{s+1/p-2}^p(\Omega)^n,$$

$$(10.25) \qquad \partial_\nu^\lambda(\vec{u}, \pi)_{\vec{f}-\mu \vec{u}} = 0 \quad \text{in } B_{s-1}^{p,p}(\partial\Omega)^n,$$

$$(10.26) \qquad \text{and } (\mu \widetilde{I} + \widetilde{B}_\lambda) \vec{u} = \widehat{\mathbb{P}}_{s,p} \vec{f} \quad \text{in } \left( V^{2-s,p'}(\Omega) \right)^*.$$

The stage has now been set for us to prove the following.

**Theorem 10.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a Lipschitz domain and assume that  $\lambda \in (-1, 1]$ . Then the domain of the fractional power of the Neumann-Stokes operator  $B_\lambda$  satisfies*

$$(10.27) \qquad D(B_\lambda^{\frac{s}{2}}) = \left\{ \vec{u} \in L_s^2(\Omega)^n : \text{div } \vec{u} = 0 \quad \text{in } \Omega \right\} \quad \text{if } s \in (1, \frac{3}{2}).$$

Furthermore, corresponding to  $s = 3/2$ , one has that  $\vec{u} \in D(B_\lambda^{\frac{3}{4}})$  if and only if

$$(10.28) \quad \begin{cases} \vec{u} \in \mathcal{V} \text{ and } \exists \pi \in L^2(\Omega), \exists \vec{f} \in L^2_{-1/2,0}(\Omega)^n \hookrightarrow L^2_{-1,0}(\Omega)^n, \\ \text{such that } (1 - \Delta) \vec{u} - \nabla \pi = \vec{f} \Big|_{\Omega} \text{ in } L^2_{-1/2}(\Omega)^n \hookrightarrow L^2_{-1}(\Omega)^n, \\ \text{and for which } \partial_{\nu}^{\lambda}(\vec{u}, \pi)_{\vec{f}-\vec{u}} = 0 \text{ in } L^2_{-1/2}(\partial\Omega)^n. \end{cases}$$

*Proof.* Assume that  $s \in [1, 2]$  and recall (9.5). Much as with (9.6), we have

$$(10.29) \quad \vec{u} \in D(B_{\lambda}^{\frac{s}{2}}) \iff \vec{u} \in \mathcal{V} \text{ and } (\tilde{I} + \tilde{B}_{\lambda})\vec{u} \in \left(V^{2-s,2}(\Omega)\right)^* \hookrightarrow \mathcal{V}^*.$$

Now, given  $\vec{u} \in D(B_{\lambda}^{\frac{s}{2}})$ , Proposition 10.2 ensures that there exist  $\vec{f}$ ,  $\pi$  as in (10.11) such that (10.12)-(10.14) are satisfied. On the other hand, from Lemma 10.1 we know that the operator (10.1) is onto. This implies that there exists  $\vec{g} \in L^2_{s-2,0}(\Omega)^n$  such that  $\tilde{\mathbb{P}}_{1,2}\vec{g} = (\tilde{I} + \tilde{B}_{\lambda})\vec{u}$  in  $\mathcal{V}^*$ . Then, according to the second part in the statement of Proposition 10.2, there exists  $q \in L^2(\Omega)$  such that (10.15)-(10.16) hold. As a consequence, if  $\tilde{\pi} := \pi - q$ , then for each  $s \in [1, 2]$ ,

$$(10.30) \quad \vec{u} \in D(B_{\lambda}^{\frac{s}{2}}) \iff \begin{cases} \vec{u} \in \mathcal{V} \text{ and } \exists \tilde{\pi} \in L^2(\Omega), \exists \vec{g} \in L^2_{s-2,0}(\Omega)^n \hookrightarrow L^2_{-1,0}(\Omega)^n, \\ \text{such that } (1 - \Delta) \vec{u} - \nabla \tilde{\pi} = \vec{g} \Big|_{\Omega} \text{ in } L^2_{s-2}(\Omega)^n \hookrightarrow L^2_{-1}(\Omega)^n, \\ \text{and for which } \partial_{\nu}^{\lambda}(\vec{u}, \tilde{\pi})_{\vec{g}-\vec{u}} = 0 \text{ in } L^2_{-1/2}(\partial\Omega)^n. \end{cases}$$

After adjusting notation, this equivalence with  $s = 3/2$  proves (10.28).

Assume next that  $s \in (1, \frac{3}{2})$ . With  $\vec{u}$ ,  $\tilde{\pi}$  and  $\vec{g}$  as in the right-hand side of (10.30), let  $(\vec{w}, \rho)$  solve

$$(10.31) \quad \begin{cases} \vec{u} \in L^2_s(\Omega)^n, \rho \in L^2_{s-1}(\Omega), \\ (1 - \Delta) \vec{w} - \nabla \rho = \vec{g} \Big|_{\Omega}, \\ \operatorname{div} \vec{w} = 0 \text{ in } \Omega, \\ \partial_{\nu}^{\lambda}(\vec{w}, \rho)_{\vec{g}-\vec{w}} = 0 \text{ in } L^2_{s-3/2}(\partial\Omega)^n. \end{cases}$$

That this is possible is ensured by Theorem 8.1. Then the difference  $(\vec{v}, \eta) := (\vec{u}, \tilde{\pi}) - (\vec{w}, \rho)$  solves the homogeneous system

$$(10.32) \quad \begin{cases} \vec{v} \in L^2_s(\Omega)^n, \eta \in L^2_{s-1}(\Omega), \\ (1 - \Delta) \vec{v} - \nabla \eta = 0 \text{ in } \Omega, \\ \operatorname{div} \vec{v} = 0 \text{ in } \Omega, \\ \partial_{\nu}^{\lambda}(\vec{v}, \eta)_{-\vec{v}} = 0 \text{ in } L^2_{-1/2}(\partial\Omega)^n. \end{cases}$$

This then forces  $\partial_{\nu}^{\lambda}(\vec{v}, \eta)_{-\vec{v}} = 0$  in  $L^2_{s-3/2}(\partial\Omega)^n$  and, hence,  $\vec{v} = 0$ ,  $\eta = 0$  in  $\Omega$  by the uniqueness part in Theorem 8.1. Thus, ultimately,  $\vec{u} = \vec{w} \in L^2_s(\Omega)^n$  and  $\tilde{\pi} = \rho \in L^2_{s-1}(\Omega)$ .

This proves the left-to-right inclusion in (10.27). The opposite implication in (10.27) then follows from (10.30) and Proposition 10.3 (considered with  $p = 2$  and  $\mu = 1$ ).  $\square$

Having established Theorem 10.4, the same argument as in the proof of Theorem 9.2 yields the following:

**Corollary 10.5.** *The end-point case  $s = 3/2$  in (9.9) holds as well. As a corollary, if  $n = 3$  then*

$$(10.33) \quad D(B_\lambda^{\frac{3}{4}}) \subset L_1^3(\Omega)^3.$$

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