Optimal estimates for the inhomogeneous problem for the bi-Laplacian in three-dimensional Lipschitz domains *

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Abstract

We establish the well-posedness of the inhomogeneous Dirichlet problem for Δ^2 in arbitrary Lipschitz domains in $\mathbb{R}^3$, with data from Besov-Triebel-Lizorkin spaces, for the optimal range of indices. The main novel contribution is to allow for certain non-locally convex spaces to be considered, and to establish integral representations for the solution.

Contents

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1 Introduction

Denote by $B^{p,q}_s$ and $F^{p,q}_s$ the Besov and Triebel-Lizorkin spaces – considered here either in a domain Ω, or on its boundary – and let Tr stand for the boundary trace operator. The main result of this paper reads as follows.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain (of arbitrary topology) with unit normal $\nu = (\nu_j)_j$. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the following property. Suppose that $0 < q \leq \infty$ and that s, p are such that either of the following two conditions*

$$\begin{aligned} &(I): \quad 0 \leq \tfrac{1}{p} < \tfrac{s}{2} + \tfrac{1+\varepsilon}{2} \quad \textit{and} \quad 0 < s < \varepsilon, \\ &(II): \quad -\tfrac{\varepsilon}{2} < \tfrac{1}{p} - \tfrac{s}{2} < \tfrac{1+\varepsilon}{2} \quad \textit{and} \quad \varepsilon \leq s < 1, \end{aligned} \tag{1.1}$$

holds. Then the problem

$$\begin{cases} \Delta^2 u = f \in B^{p,q}_{s+\frac{1}{p}-3}(\Omega), \\ \operatorname{Tr} u = f_0 \in B^{p,q}_s(\partial\Omega), \\ \operatorname{Tr}(\partial_j u) = f_j \in B^{p,q}_s(\partial\Omega), \;\; 1 \leq j \leq 3, \end{cases} \tag{1.2}$$

where the boundary data satisfy the (necessary) compatibility conditions

$$(\nu_j \partial_k - \nu_k \partial_j) f_0 = \nu_j f_k - \nu_k f_j, \qquad \forall\, j, k, \tag{1.3}$$

has a unique solution $u \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega)$. This satisfies

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}+1}(\Omega)} \leq C\Big(\|f\|_{B^{p,q}_{s+\frac{1}{p}-3}(\Omega)} + \sum_{j=0}^{3} \|f_j\|_{B^{p,q}_s(\partial\Omega)} \Big), \tag{1.4}$$

for some finite constant $C = C(\Omega, s, p) > 0$.

Furthermore, similar results are valid for the version of the above boundary problem phrased on Triebel-Lizorkin spaces, that is, for

$$\begin{cases} \Delta^2 u = f \in F^{p,q}_{s+\frac{1}{p}-3}(\Omega), \\ \operatorname{Tr} u = f_0 \in B^{p,p}_s(\partial\Omega), \\ \operatorname{Tr}(\partial_j u) = f_j \in B^{p,p}_s(\partial\Omega), \;\; 1 \leq j \leq 3, \end{cases} \tag{1.5}$$

where $u \in F^{p,q}_{s+\frac{1}{p}+1}(\Omega)$ and the boundary data is assumed to satisfy the compatibility conditions (1.3) (and $p, q < \infty$).

It is easy to see that f_0 and $(f_j)_j$ are necessarily related if (1.2) has a solution, for in that case $\nabla_{tan} f_0 = \nabla u - (\partial_\nu u)\nu = (f_j - \nu_j \nu_k f_k)_j$. The reason we prefer to state this as in (1.3) has to do with the fact that, as opposed to the tangential gradient $\nabla_{tan} f_0$, the tangential derivative operators $\partial_{\tau_{jk}} := \nu_j \partial_k - \nu_k \partial_j$ are well-defined when acting from $B^{p,q}_s(\partial\Omega)$ into $B^{p,q}_{s-1}(\partial\Omega)$ (if $0 < p, q \leq \infty$, $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$) even if Ω is merely Lipschitz.

Theorem 1.1 refines the well-posedness results for Lipschitz domains in $\mathbb{R}^3$ obtained by V. Adolfsson and J. Pipher in [1] where the authors treat the case of (1.5) (with homogeneous boundary conditions) in the situation when, in addition to having either of the two conditions in (1.1) satisfied, one also asks that $1 < p < \infty$ and $q = 2$. In addition, we are able to obtain integral representations for the solutions of (1.2) and (1.5). Somewhat more specifically, if $1 < q < \infty$ and $(s, 1/p) \in (0,1) \times (0,1)$ are such that one of the conditions in (1.1) is satisfied, then the solution u of the problem (1.2) with $f = 0$ can be written in the form

$$u(X) = \int_{\partial\Omega} |X - Y| g_0(Y)\, d\sigma(Y) + \sum_{j=1}^{3} \int_{\partial\Omega} \frac{x_j - y_j}{|X - Y|} g_j(Y)\, d\sigma(Y), \quad X \in \Omega, \tag{1.6}$$

for some functions $g_j \in B^{p,q}_{s-1}(\partial\Omega)$, $0 \leq j \leq 3$. A similar representation is valid for the solution of (1.5). See Theorem 5.10 for a more precise result.

The work of V. Adolfsson and J. Pipher in [1] also deals with the case of the problem (1.5) (with homogeneous boundary conditions) considered in bounded C^1 domains in $\mathbb{R}^n$, $n \geq 3$, in which setting the authors prove its well-posedness for $q = 2$ and all $1 < p < \infty$, $0 < s < 1$. This latter scenario is, nonetheless, covered by the following result proven by I. Mitrea and M. Mitrea in [36], by further building on the work of V. Maz'ya, M. Mitrea and T. Shaposhnikova in [31]:

Theorem 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, whose outward unit normal vector ν belongs to $VMO(\partial\Omega)$, the Sarason space of functions of vanishing mean oscillations on $\partial\Omega$. Then the problem*

$$\begin{cases} \Delta^2 u = f \in F^{p,q}_{s+\frac{1}{p}-3}(\Omega) \quad \text{in } \Omega, \\ \operatorname{Tr} u = f_0 \in B^{p,p}_s(\partial\Omega), \\ \operatorname{Tr}(\partial_j u) = f_j \in B^{p,p}_s(\partial\Omega), \ \ 1 \leq j \leq n, \\ u \in F^{p,q}_{1+s+\frac{1}{p}}(\Omega), \end{cases} \tag{1.7}$$

with $f_0, f_1, ..., f_n$ satisfying compatibility conditions (1.3), is well-posed whenever

$$0 < s < 1, \qquad 1 < p < \infty, \qquad 0 < q \leq \infty. \tag{1.8}$$

The result in [36] is actually more general, in that Δ^2 can be replaced by an arbitrary elliptic system of homogeneous, constant coefficient differential operators, and the domain in question could even be Lipschitz. Then whether a given triplet of exponents (p, q, s) leads to well-posed problem depends on the size of dist $(\nu, VMO(\partial\Omega))$, where the distance is measured in $BMO(\partial\Omega)$. Here $BMO(\partial\Omega)$ is the John-Nirenberg space of functions bounded mean oscillations, and $VMO(\partial\Omega)$ is the Sarason space of functions of vanishing mean oscillations. Thus, in general, $C^0(\partial\Omega) \subset VMO(\partial\Omega) \subset BMO(\partial\Omega)$, and $\nu \in L^\infty(\partial\Omega) \subset BMO(\partial\Omega)$. In particular, $\partial\Omega \in C^1$ entails $\mathrm{dist}\,(\nu, VMO(\partial\Omega)) = 0$.

The consideration of the full scales of Triebel-Lizorkin spaces $F^{p,q}_\alpha(\Omega)$ and Besov spaces $B^{p,q}_\alpha(\Omega)$ is both natural and utilitarian. Indeed, as is well-known, these scales encompass a number of more specialized function spaces which appear frequently in practical applications. Below, we wish to elaborate more on this aspect, in a fashion which emphasizes the smoothing properties of the Green operator $\mathbf{G}$ for the inhomogeneous Dirichlet problem for the bi-Laplacian. That is, formally, if u solves

$$\Delta^2 u = f \quad \text{in} \quad \Omega, \qquad u = \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.9}$$

then

$$\mathbf{G}f := u. \tag{1.10}$$

These considerations can be made precise using Lax-Milgram's lemma, ultimately yielding that

$$\mathbf{G} : W^{-2,2}(\Omega) \longrightarrow \overset{\circ}{W}{}^{2,2}(\Omega) \qquad \text{boundedly}, \tag{1.11}$$

where $\overset{\circ}{W}{}^{2,2}(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in the norm $\|\Delta u\|_{L^2(\Omega)} \approx \sum_{|\alpha|\leq 2} \|\partial^\alpha u\|_{L^2(\Omega)}$, and $W^{-2,2}(\Omega) := \{\sum_{|\alpha|\leq 2} \partial^\alpha u_\alpha : u_\alpha \in L^2(\Omega)\}$ equipped with the natural infimum norm. Hence, the issue at stake is replacing the L^2-based Sobolev spaces by more general Besov and Triebel-Lizorkin spaces. Set $B^{p,q}_{\alpha,z}(\Omega) := \{f|_\Omega : f \in B^{p,q}_\alpha(\mathbb{R}^n),\ \mathrm{supp}\, f \subseteq \bar{\Omega}\}$, with a similar definition for $F^{p,q}_{\alpha,z}(\Omega)$. Based on Theorem 1.1 and trace results, we can then show the following.

Corollary 1.3. *For every bounded, Lipschitz domain Ω in $\mathbb{R}^3$ there exists some small number $\varepsilon = \varepsilon(\Omega) > 0$ such the operators*

$$\mathbf{G} : B^{p,q}_{\alpha-1}(\Omega) \longrightarrow B^{p,q}_{\alpha+3,z}(\Omega), \tag{1.12}$$

$$\mathbf{G} : F^{p,q}_{\alpha-1}(\Omega) \longrightarrow F^{p,q}_{\alpha+3,z}(\Omega), \tag{1.13}$$

are isomorphisms whenever $0 < q \leq \infty$ for the Besov scale, and $\min\{p, 1\} \leq q < \infty$ for the Triebel-Lizorkin scale, and the point with coordinates $(\alpha, 1/p)$ belongs to the (open) pentagonal region depicted in Figure 1 below.

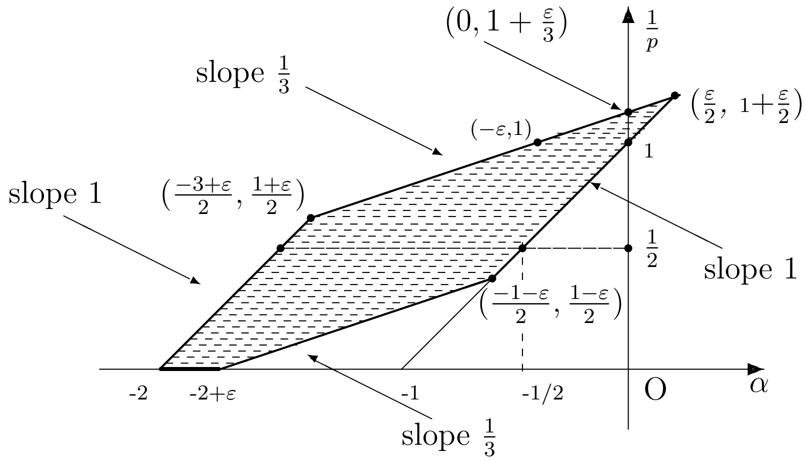

Figure 1

Also,

$$\nabla^4 \mathbf{G} : B^{p,q}_{\alpha-1}(\Omega) \longrightarrow B^{p,q}_{\alpha-1}(\Omega), \tag{1.14}$$

$$\nabla^4 \mathbf{G} : F^{p,q}_{\alpha-1}(\Omega) \longrightarrow F^{p,q}_{\alpha-1}(\Omega), \tag{1.15}$$

are bounded operators whenever $0 < q \leq \infty$ *and the point with coordinates* $(\alpha, 1/p)$ *belongs to the (open) pentagonal region from Figure 1.*

A remarkable special case corresponds to the Triebel-Lizorkin scale with $q = 2$ and $\alpha = 0$, in which case one obtains the following.

Corollary 1.4. *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz. Then there exists* $\varepsilon = \varepsilon(\Omega) > 0$ *such that*

$$\nabla^4 \mathbf{G} : F^{p,2}_{-1}(\Omega) \longrightarrow F^{p,2}_{-1}(\Omega) \tag{1.16}$$

is a bounded operator provided that $1 - \varepsilon < p < 1$.

Recall that if $\frac{n}{n+1} < p \leq 1$ then $F^{p,2}_0(\Omega)$ becomes $h^p(\Omega) := \{f|_\Omega : f \in h^p(\mathbb{R}^n)\}$, where $h^p(\mathbb{R}^n)$ is the local Hardy space in $\mathbb{R}^n$. Hence, $F^{p,2}_{-1}(\Omega)$ can be thought of as the space of (at most) first order derivatives of distributions from $h^p(\Omega)$. For the Laplace operator, a similar result (valid in all space dimensions) has been established by S. Mayboroda and M. Mitrea in [30]. This answered in the affirmative a conjecture made by D.-C. Chang, S. Krantz and E. Stein (cf. [5], [6]) regarding the regularity of the harmonic Green potentials on Hardy spaces in Lipschitz domains. The corresponding analogue of this result for the Stokes system for arbitrary Lipschitz domains in dimension 3 has been recently proved by M. Mitrea and M. Wright in [39].

Regarding the off-diagonal Besov scale, here we wish to single out the following.

Corollary 1.5. *For every bounded Lipschitz domain Ω in $\mathbb{R}^3$, the operator*

$$\mathbf{G} : B^{p,1}_{\frac{3}{p}-3}(\Omega) \longrightarrow C^1(\overline{\Omega}), \qquad p \in (0,\infty), \tag{1.17}$$

is well-defined, and

$$\|\nabla \mathbf{G} f\|_{L^\infty(\Omega)} \leq C(\Omega,p)\,\|f\|_{B^{p,1}_{\frac{3}{p}-3}(\Omega)}, \qquad 0<p<\infty. \tag{1.18}$$

This can be regarded as the inhomogeneous analogue of the Maximum Principle established for the bi-Laplacian in Lipschitz domains by J. Pipher and G. Verchota in [44] (cf. also [45]). This (essentially) states that for any bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$ there holds

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^\infty(\partial\Omega)}, \tag{1.19}$$

uniformly for u biharmonic in Ω, and in fact, (1.18) can also be derived from (1.19). Other related L^∞-estimates are as follows. When specialized to the case of the bi-Laplacian, Theorem 3.3 on p. 329 of J. Nečas' book [41] yields that if $\Omega\subset\mathbb{R}^3$ satisfies a uniform exterior ball condition then

$$\nabla \mathbf{G} : F^{2,2}_{-1}(\Omega) \longrightarrow L^\infty(\Omega). \tag{1.20}$$

See also J. Seo's paper [50] for some related estimates. More recently, S. Mayboroda and V. Maz'ya have proved in [28] that if $\Omega\subset\mathbb{R}^3$ is bounded and open then

$$\nabla \mathbf{G} : L^{\frac{3}{2},1}_{-1}(\Omega) \longrightarrow L^\infty(\Omega), \tag{1.21}$$

where $L^{p,q}_{-1}(\Omega)$ consists of first order derivatives of functions from the Lorentz space $L^{p,q}(\Omega)$. Note that $F^{2,2}_{-1}(\Omega) \hookrightarrow L^{3/2,1}_{-1}(\Omega) \hookrightarrow B^{p,1}_{3/p-3}(\Omega)$ for any $p>\frac{3}{2}$ (cf. (4.50) for the last embedding). (Parenthetically, we note that another interesting strictly smaller subspace of $B^{p,1}_{3/p-3}(\Omega)$, when $p>1$, is the local Hardy space $h^1(\Omega)$).) Moreover, as pointed out by S. Mayboroda and V. Maz'ya in [28], it is not generally the case that $\nabla\mathbf{G}f$ is continuous on $\bar{\Omega}$ for each $f\in L^{3/2,1}_{-1}(\Omega)$ if Ω lacks any smoothness.

To place this work in the proper perspective, let us recall some of the known positive results and counterexamples in the case of the Laplacian. In their paper [19] on the inhomogeneous problem for the Laplacian in Lipschitz domains, D. Jerison and C. Kenig have obtained well-posedness results on Sobolev spaces and the (diagonal) Besov scale $B^{p,p}_\alpha$ with $1\leq p\leq\infty$. With $\mathscr{G}$ denoting the Green operator associated with the Dirichlet Laplacian in the Lipschitz domain $\Omega\subset\mathbb{R}^n$, they have shown that

$$\nabla^2\mathscr{G} : B^{p,p}_\alpha(\Omega) \to B^{p,p}_\alpha(\Omega) \tag{1.22}$$

boundedly whenever $(\alpha, 1/p)$ belongs to the subregion in Figure 1 corresponding to $p \geq 1$. The counterexamples given by D. Jerison and C. Kenig in [19] show that this subregion is optimal, but only if one insists that $p \geq 1$ (when all spaces involved are Banach). However, the Besov scale $B^{p,p}_\alpha$ naturally continues below $p = 1$, though the corresponding spaces are no longer locally convex. The consideration of the entire scales $B^{p,q}_\alpha$, $F^{p,q}_\alpha$, $0 < p, q < \infty$, is also natural because Hardy spaces occur precisely when $p \leq 1$ on the Triebel-Lizorkin scale, and because Besov spaces with $p < 1$ offer a natural framework for certain types of numerical approximation schemes.

When $\partial\Omega \in C^\infty$ the operator $\partial_j\partial_k\mathscr{G}$ falls under the scope of the classical theory of singular integral operators of Calderón-Zygmund type. In particular, it maps $L^p(\Omega)$ boundedly into itself for any $1 < p < \infty$ –this is the point of view adopted by S. Agmon, A. Douglis and L. Nirenberg in [2]. In fact, as alluded to before, it was proved by D. Chang, G. Dafni and E. Stein in [7] that $\nabla^2\mathscr{G}$ also maps $h^p(\Omega)$ boundedly into itself if $\frac{n}{n+1} < p \leq 1$, provided $\partial\Omega \in C^\infty$. More general results of this type have been proved by J. Franke and T. Runst in [18]. When specialized to the case of the bi-Laplacian, these imply that

$$\partial\Omega \in C^\infty \Longrightarrow \left\{ \begin{array}{l} \text{problem (1.2) has a unique solution } u \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega) \\ \text{whenever } 0 < p, q \leq \infty \text{ and } s > (n-1)\big(\frac{1}{p} - 1\big)_+, \end{array} \right. \tag{1.23}$$

with a similar result on the Triebel-Lizorkin scale. Furthermore, simple counterexamples show that the condition $s \geq (n-1)\big(\frac{1}{p} - 1\big)_+$ is necessary for the well-posedness of (1.2) even when $\partial\Omega$ is C^∞. The situation is drastically different in less smooth domains. For example, B. Dahlberg has constructed in [10] a bounded C^1-domain Ω along with a function $f \in C^\infty(\bar{\Omega})$, such that $\partial_j\partial_k\mathscr{G} f \notin L^p(\Omega)$ for any $p > 1$.

The methods developed by D. Jerison and C. Kenig in [19], although beautiful in their elegance and sharpness, rely in an essential fashion on the maximum principle and, as such, do not readily adapt to other natural boundary conditions, e.g., of Neumann type. In fact, the latter issue was singled out as open problem # 3.2.21 in Kenig's book [26]. Subsequently, this has been solved by E. Fabes, O. Mendez and M. Mitrea in [16] –cf. also the work of M. Mitrea and M. Taylor in [38]– via a new approach which relies on a systematic use of singular integral operators. However, the works just cited still assumed $p \geq 1$. The fact that (1.22) actually holds for $(\alpha, 1/p)$ belonging to the region depicted in Figure 1 has been shown in [30], using a blend of techniques from harmonic and functional analysis. From this perspective, it was of importance that the solutions had integral representations. As already mentioned, the case of the bi-Laplacian has been dealt with by V. Adolfsson and J. Pipher in [1], albeit working directly with the problem, without providing integral representations for solutions.

The present work makes heavy use of singular integral operators of Calderón-Zygmund type, atomic estimates, and (functional analytic) perturbation theory. In particular, it is strongly influenced by the earlier work of J. Pipher and G. Verchota in [43], whose key achievement is an H^1 atomic estimate in graph domains. For our current purposes we need to rework the results in [43] in bounded star-like Lipschitz domains and with integral representations. It should be noted that there are significant technical differences between these two cases. In particular, we find the correct formulation of the "non-standard" regularity boundary value problem, then solve it, via singular integral operators.

Along the way, we deal with a number of related problems for Δ^2, such as the Dirichlet problem in a bounded Lipschitz domain Ω in $\mathbb{R}^n$ with boundary data from:

(i) $L^p(\partial\Omega)$ provided $2-\varepsilon<p<2+\varepsilon$, for $n\geq 3$;

(ii) $L^p_1(\partial\Omega)$ provided $2-\varepsilon<p<2+\varepsilon$, for $n\geq 3$;

(iii) $L^p(\partial\Omega)$ if $2-\varepsilon<p<\infty$, $C^s(\partial\Omega)$ if $0<s<\varepsilon$, $\mathrm{bmo}\,(\partial\Omega)$, or $\mathrm{vmo}\,(\partial\Omega)$, for $n=3$;

(iv) $L^p_1(\partial\Omega)$ if $1<p<2+\varepsilon$, or $h^{1,p}_{at}(\partial\Omega)$ if $1-\varepsilon<p\leq 1$, for $n=3$,

where $\varepsilon=\varepsilon(\Omega)>0$. We also formulate and prove the well-posedness of the corresponding exterior versions of these problems. Some of these results have already been treated in the literature, albeit via a somewhat different approach, while others are new. See the work of B. Dahlberg, C. Kenig and G. Verchota in [13], of J. Pipher and G. Verchota in [43], [44]. Related work is by J. Pipher and G. Verchota in [42], [45], by Z. Shen in [51], [52], by V. Maz'ya, S. Nazarov and B. Plamenevskii in [32], [33], by G. Verchota in [56], [57], [58], by I. Mitrea and M. Mitrea in [36].

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2 Singular integrals in star-like Lipschitz domains

2.1 The geometry of star-like Lipschitz domains

Call a bounded open set $\Omega\subset\mathbb{R}^n$ a bounded Lipschitz domain if there exists a finite open covering $\{\mathcal{O}_j\}_{1\leq j\leq N}$ of $\partial\Omega$ with the property that, for every $j\in\{1,...,N\}$, $\mathcal{O}_j\cap\Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j:\mathbb{R}^{n-1}\to\mathbb{R}$ (where $\mathbb{R}^{n-1}\times\mathbb{R}$ is a new system of coordinates obtained from the original one via a rigid motion). As is well-known, for a Lipschitz domain Ω, the surface measure $d\sigma$ is well-defined on $\partial\Omega$ and there exists an outward pointing normal vector $\nu=(\nu_1,\dots,\nu_n)$ at almost every point on $\partial\Omega$.

Given $\alpha>0$, we set

$$\Gamma_\alpha(X):=\{Y\in\Omega:\,|X-Y|\leq(1+\alpha)\,\mathrm{dist}\,(Y,\partial\Omega)\}\tag{2.1}$$

and denote by $M=M_\alpha$ the nontangential maximal operator associated with Ω. That is, for a function u defined in Ω, set

$$(Mu)(X) := \sup_{Y \in \Gamma_\alpha(X)} |u(Y)|, \qquad X \in \partial\Omega. \tag{2.2}$$

Also, define the nontangential pointwise trace by

$$u\Big|_{\partial\Omega}(X) := \lim_{\substack{Y \in \Gamma_\alpha(X) \\ Y \to X}} u(Y), \qquad X \in \partial\Omega. \tag{2.3}$$

Finally, adopt a similar definition for $u|_{\partial\Omega_-}$, in the case when u is defined in $\mathbb{R}^n \setminus \bar{\Omega}$.

Assume next that $n \geq 2$ and that

$$\begin{aligned} &\Omega \subset \mathbb{R}^n \text{ is a bounded Lipschitz domain,} \\ &\text{which is starlike with respect to the origin.} \end{aligned} \tag{2.4}$$

Then there exists

$$\varphi : S^{n-1} \longrightarrow (0, \infty), \quad \text{Lipschitz}, \tag{2.5}$$

(where $S^{n-1} = \partial B(0,1)$ is the unit sphere in $\mathbb{R}^n$, centered at the origin) such that, in polar coordinates,

$$\Omega = \{r\omega : \omega \in S^{n-1},\ 0 \leq r < \varphi(\omega)\}. \tag{2.6}$$

In particular,

$$\partial\Omega = \{\varphi(\omega)\omega : \omega \in S^{n-1}\}, \tag{2.7}$$

the outward unit normal to $\partial\Omega$, $\nu = (\nu_1, ..., \nu_n)$, is given by

$$\nu(\varphi(\omega)\omega) = \frac{\varphi(\omega)\omega - (\nabla_{tan}\varphi)(\omega)}{\sqrt{|(\nabla_{tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}}, \qquad \text{for a.e. } \omega \in S^{n-1}, \tag{2.8}$$

(where $\nabla_{tan}\varphi$ denotes the tangential gradient of φ on S^{n-1}), and surface measure σ on $\partial\Omega$ satisfies

$$\int_{\partial\Omega} f\, d\sigma = \int_{S^{n-1}} f(\omega\varphi(\omega))[\varphi(\omega)]^{n-2}\sqrt{|(\nabla_{tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}\, d\omega, \tag{2.9}$$

for any absolutely integrable function f on $\partial\Omega$. See [21] for a proof. Here we only wish to mention that we define the normalized Lipschitz constant of φ in (2.5) as

$$\mathrm{Lip}\,(\varphi) := \left(\sup_{\omega,\omega'\in S^{n-1}} \frac{|\varphi(\omega)-\varphi(\omega')|}{|\omega-\omega'|}\right)\left(\inf_{\omega\in S^{n-1}}\varphi(\omega)\right)^{-1}. \tag{2.10}$$

Note that $\mathrm{Lip}\,(\varphi)$ is invariant under dilations of Ω in $\mathbb{R}^n$. Throughout the paper, we set

$$\Omega_+ := \Omega, \qquad \Omega_- := \mathbb{R}^n\setminus\overline{\Omega}, \tag{2.11}$$

(hence, the outward unit normal to Ω_- is $-\nu$) and define

$$\eta(X) := X, \qquad X\in\mathbb{R}^n. \tag{2.12}$$

For a function $f:\partial\Omega\to\mathbb{R}$, denote by $\tilde{f}$ the extension of (the pull-back to S^{n-1} of) f as a function homogeneous of degree zero in $\mathbb{R}^n\setminus\{0\}$. Specifically, we define

$$\tilde{f}:\mathbb{R}^n\setminus\{0\}\longrightarrow\mathbb{R}, \qquad \tilde{f}(X) := f\Big(\varphi\Big(\tfrac{X}{|X|}\Big)\tfrac{X}{|X|}\Big). \tag{2.13}$$

Then, with $\langle\cdot,\cdot\rangle$ denoting the canonical inner product of vectors in $\mathbb{R}^n$, a direct calculation shows that

$$\begin{aligned}
(\partial_j\tilde{f})(X) &= (\partial_j f)\Big(\varphi\Big(\tfrac{X}{|X|}\Big)\tfrac{X}{|X|}\Big)\varphi\Big(\tfrac{X}{|X|}\Big)\tfrac{1}{|X|}\\
&\quad+\Big\langle\tfrac{X}{|X|}\,,\,(\nabla f)\Big(\tfrac{X}{|X|}\Big)\Big\rangle\tfrac{1}{|X|}\Big((\nabla_{tan}\varphi)\Big(\tfrac{X}{|X|}\Big)-\varphi\Big(\tfrac{X}{|X|}\Big)\Big)_j\\
&= \varphi\Big(\tfrac{X}{|X|}\Big)\tfrac{1}{|X|}\Big\langle(\nabla f)\Big(\tfrac{X}{|X|}\Big)\,,\,e_j-\nu_j\Big(\varphi\Big(\tfrac{X}{|X|}\Big)\tfrac{X}{|X|}\Big)\Big/\Big\langle\tfrac{X}{|X|},\nu\Big(\varphi\Big(\tfrac{X}{|X|}\Big)\tfrac{X}{|X|}\Big)\Big\rangle\Big\rangle.
\end{aligned} \tag{2.14}$$

Specializing this to the case when $X=\varphi(\omega)\omega$ for some $\omega\in S^{n-1}$, we obtain

$$\begin{aligned}
(\partial_j\tilde{f})(X) &= \Big\langle(\nabla f)(X)\,,\,e_j-\frac{\nu_j(X)}{\langle X,\nu(X)\rangle}X\Big\rangle\\
&= \Big\langle(\nabla_{tan} f)(X)\,,\,e_j-\frac{\nu_j(X)}{\langle X,\nu(X)\rangle}X\Big\rangle, \qquad X\in\partial\Omega,
\end{aligned} \tag{2.15}$$

since, for each $j\in\{1,...,n\}$, the vector field

$$\vec{T}_j(X) := e_j-\frac{\nu_j(X)}{\langle X,\nu(X)\rangle}X, \qquad X\in\partial\Omega, \tag{2.16}$$

is tangential to $\partial\Omega$. Above, $e_j := (\delta_{jk})_{1\le k\le n}$ stands for the j-th vector in the standard orthonormal basis in $\mathbb{R}^n$, and $\nabla_{tan} f$ denotes the tangential gradient of f on $\partial\Omega$, i.e.

$$\nabla_{tan} f := \Big(\nu_k \partial_{\tau_{kj}} f\Big)_{1\leq j\leq n}, \tag{2.17}$$

with the summation convention over repeated indices understood. Here and elsewhere, $\partial_{\tau_{jk}}$ denotes the tangential derivative

$$\partial_{\tau_{jk}} = \nu_j \partial_k - \nu_k \partial_j = \vec{T}_{jk} \cdot \nabla, \quad \vec{T}_{jk} := \nu_j e_k - \nu_k e_j, \quad j,k \in \{1,...,n\}. \tag{2.18}$$

In summary, with $\eta(X) := X$,

$$\partial_j \tilde{f} = \Big\langle \nabla_{tan} f \, , \, e_j - \frac{\nu_j}{\langle \eta, \nu\rangle} \, \eta \Big\rangle = \langle \nabla_{tan} f, \vec{T}_j \rangle, \qquad \text{on } \ \partial\Omega. \tag{2.19}$$

Going further, call a family of $n+1$ functions defined on $\partial\Omega$,

$$\dot{f} = (f_0, f_1, ..., f_n), \tag{2.20}$$

a Whitney array provided the following compatibility conditions are satisfied:

$$\partial_{\tau_{jk}} f_0 = \nu_j f_k - \nu_k f_j, \qquad j,k \in \{1,...,n\}. \tag{2.21}$$

Assuming that this the case, we compute

$$\begin{aligned} \partial_j \widetilde{f_0} \ &= \ \Big\langle \nabla_{tan} f_0 \, , \, e_j - \frac{\nu_j}{\langle \eta, \nu\rangle} \, \eta \Big\rangle = \nu_k (\partial_{\tau_{kr}} f_0) \Big(\delta_{jr} - \frac{\nu_j \eta_r}{\langle \eta, \nu\rangle}\Big) \\ &= \ (f_r - \nu_k \nu_r f_k) \Big(\delta_{jr} - \frac{\nu_j \eta_r}{\langle \eta, \nu\rangle}\Big) = f_j - \frac{\nu_j}{\langle \eta, \nu\rangle} \eta_k f_k, \end{aligned} \tag{2.22}$$

i.e.,

$$\partial_j \widetilde{f_0} = f_j - \frac{\nu_j}{\langle \eta, \nu\rangle} \eta_k f_k \qquad \text{on } \ \partial\Omega. \tag{2.23}$$

For further reference, let us also remark the following. Define by $\Delta_{S^{n-1}}$ the Laplace-Beltrami operator on S^{n-1}. Then, in polar coordinates,

$$\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}}. \tag{2.24}$$

Given $f, g : S^{n-1} \to \mathbb{R}$, we also have

$$-\int_{S^{n-1}} (\Delta_{S^{n-1}} f)(\omega) g(\omega)\, d\omega \;=\; \int_{S^{n-1}} \langle \nabla_{tan} f(\omega), \nabla_{tan} g(\omega) \rangle\, d\omega \qquad (2.25)$$

$$= \int_{S^{n-1}} \langle (\nabla f_{\#})(\omega), (\nabla g_{\#})(\omega) \rangle\, d\omega,$$

where, in this context, ∇_{tan} denotes the tangential gradient on S^{n-1}, and we have set $f_{\#}(X) := f(X/|X|)$ for $X \in \mathbb{R}^n \setminus \{0\}$. In particular,

$$-\int_{S^{n-1}} (\Delta_{S^{n-1}} f)(\omega) f(\omega)\, d\omega = \int_{S^{n-1}} |\nabla_{tan} f(\omega)|^2\, d\omega, \qquad (2.26)$$

so that

$$\Delta_{S^{n-1}} f = 0 \text{ on } S^{n-1} \iff f \equiv \text{constant on } S^{n-1}. \qquad (2.27)$$

2.2 The radial derivative

Define the radial derivative of a given function u as

$$(\nabla_\eta u)(X) := x_j \partial_j u(X) = \langle X, \nabla u(X) \rangle, \qquad (2.28)$$

and note that we have the commutator identity

$$\Big[\nabla_\eta, \partial_k\Big] = -\partial_k, \qquad 1 \le k \le n. \qquad (2.29)$$

As a result,

$$\Big[\Delta, \nabla_\eta\Big] = 2\Delta, \qquad (2.30)$$

i.e.,

$$\Delta(\nabla_\eta u) = 2\Delta u + \nabla_\eta(\Delta u). \qquad (2.31)$$

In particular,

$$u \text{ harmonic} \implies \nabla_\eta u \text{ harmonic}. \qquad (2.32)$$

Also, for any numbers $0 \le t_0 < t_1 < \infty$ and any point X,

$$\int_{t_0}^{t_1} (\nabla_\eta u)(tX)\,\frac{dt}{t} = \int_{t_0}^{t_1} \langle tX, (\nabla u)(tX)\rangle\,\frac{dt}{t} \tag{2.33}$$

$$= \int_{t_0}^{t_1} \frac{d}{dt}\Big[u(tX)\Big]\,dt = u(t_1X) - u(t_0X).$$

Next, observe that if $v \in C^1$ has $v(0) = 0$, then

$$H(X) := \int_0^1 v(tX)\frac{dt}{t} \tag{2.34}$$

is well-defined (that the properties of v ensure that the integral is convergent) and

$$\nabla_\eta H(X) = \langle X, \nabla H(X)\rangle = \int_0^1 \langle X, (\nabla v)(tX)\rangle\,dt \tag{2.35}$$

$$= \int_0^1 \frac{d}{dt}\Big[v(tX)\Big]\,dt = v(X).$$

That is, H is a normalized radial anti-derivative for v, i.e.,

$$\nabla_\eta H = v \quad \text{and} \quad H(0) = 0. \tag{2.36}$$

Furthermore,

$$v \text{ harmonic} \implies H \text{ harmonic}. \tag{2.37}$$

2.3 Smoothness spaces on Lipschitz boundaries

For $a \in \mathbb{R}$ set $(a)_+ := \max\{a, 0\}$. Consider three parameters p, q, s subject to

$$0 < p, q \leq \infty, \qquad (n-1)\Big(\tfrac{1}{p} - 1\Big)_+ < s < 1 \tag{2.38}$$

and assume that $\Omega \subset \mathbb{R}^n$ is the upper-graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. We then define $B_s^{p,q}(\partial\Omega)$ as the space of locally integrable functions f on $\partial\Omega$ for which the assignment $\mathbb{R}^{n-1} \ni x \mapsto f(x, \varphi(x))$ belongs to $B_s^{p,q}(\mathbb{R}^{n-1})$, the classical Besov space in $\mathbb{R}^{n-1}$. We then define

$$\|f\|_{B_s^{p,q}(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\|_{B_s^{p,q}(\mathbb{R}^{n-1})} \tag{2.39}$$

As far as Besov spaces with a negative amount of smoothness are concerned, in the same context as above we set

$$f \in B^{p,q}_{s-1}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in B^{p,q}_{s-1}(\mathbb{R}^{n-1}), \tag{2.40}$$

$$\|f\|_{B^{p,q}_{s-1}(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2}\|_{B^{p,q}_{s-1}(\mathbb{R}^{n-1})}. \tag{2.41}$$

As is well-known, the case when $p = q = \infty$ corresponds to the usual (inhomogeneous) Hölder spaces $C^s(\partial\Omega)$, defined by the requirement that

$$\|f\|_{C^s(\partial\Omega)} := \|f\|_{L^\infty(\partial\Omega)} + \sup_{\substack{X \neq Y \\ X,Y \in \partial\Omega}} \frac{|f(X) - f(Y)|}{|X - Y|^s} < +\infty. \tag{2.42}$$

That is,

$$B^{\infty,\infty}_s(\partial\Omega) = C^s(\partial\Omega) \qquad \text{for} \quad s \in (0, 1). \tag{2.43}$$

All the above definitions then readily extend to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and fix $(n-1)/n < p < \infty$, $0 < q \leq \infty$, and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. Then, for each $j, k \in \{1, ..., n\}$, the tangential derivative operator*

$$\partial_{\tau_{jk}} : B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s-1}(\partial\Omega) \tag{2.44}$$

is well-defined, linear and bounded.

We now proceed to discuss Triebel-Lizorkin spaces defined on the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, denoted in the sequel by $F^{p,q}_s(\partial\Omega)$. Compared with the Besov scale, the most important novel aspect here is the possibility of allowing the endpoint case $s = 1$ as part of the general discussion if $q = 2$. To discuss this in more detail, assume that either

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad (n-1)\Big(\frac{1}{\min\{p, q\}} - 1\Big)_+ < s < 1, \tag{2.45}$$

or

$$\frac{n-1}{n} < p < \infty, \quad q = 2, \quad s = 1. \tag{2.46}$$

In this scenario, the Triebel-Lizorkin scale in $\mathbb{R}^{n-1}$ is invariant under pointwise multiplication by Lipschitz maps as well as composition by Lipschitz diffeomorphisms.

When Ω is a Lipschitz domain in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, we may therefore define the space $F^{p,q}_s(\partial\Omega)$ as the collection of all locally integrable functions f on $\partial\Omega$ such that

$$f(\cdot, \varphi(\cdot)) \in F^{p,q}_s(\mathbb{R}^{n-1}), \tag{2.47}$$

endowed with the norm

$$\|f\|_{F^{p,q}_s(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\|_{F^{p,q}_s(\mathbb{R}^{n-1})}. \tag{2.48}$$

Also, if $\mathrm{Lip}_0\,(\partial\Omega)$ stands for the collection of all compactly supported Lipschitz functions on $\partial\Omega$, the space $F^{p,q}_{s-1}(\partial\Omega)$ is defined as the collection of all functionals $f \in (\mathrm{Lip}_0\,(\partial\Omega))'$ such that

$$f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in F^{p,q}_{s-1}(\mathbb{R}^{n-1}), \tag{2.49}$$

and we equip with space the quasi-norm

$$\|f\|_{F^{p,q}_{s-1}(\partial\Omega)} := \|f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2}\|_{F^{p,q}_{s-1}(\mathbb{R}^{n-1})}. \tag{2.50}$$

Finally, when $\Omega \subset \mathbb{R}^n$ is a *bounded Lipschitz domain* and (s, p, q) are as in (2.45)-(2.46), we define $F^{p,q}_s(\partial\Omega)$ and $F^{p,q}_{s-1}(\partial\Omega)$ via localization (using a smooth, finite partition of unity) and pull-back to $\mathbb{R}^{n-1}$ (in the manner described above, for graph-Lipschitz domains). When equipped with the natural quasi-norms, the Triebel-Lizorkin spaces just introduced are quasi-Banach, and different partitions of unity yield equivalent quasi-norms.

Two basic identities, relating Triebel-Lizorkin spaces to Sobolev spaces on $\partial\Omega$ read as follows:

$$F^{p,2}_0(\partial\Omega) = L^p(\partial\Omega), \quad F^{p,2}_1(\partial\Omega) = L^p_1(\partial\Omega), \quad \forall\, p \in (1, \infty). \tag{2.51}$$

The second formula can be taken as a definition of the L^p-based Sobolev spaces of order one on $\partial\Omega$. For practical purposes, it is useful to point out an alternative characterization, namely

$$L^p_1(\partial\Omega) = \{f \in L^p(\partial\Omega) : \, \partial_{\tau_{jk}} f \in L^p(\partial\Omega), \; 1 \leq j, k \leq n\}, \quad p \in (1, \infty), \tag{2.52}$$

with $\|f\|_{L^p_1(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \sum_{j,k} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \|\nabla_{tan} f\|_{L^p(\partial\Omega)}$.

The above formulas have natural counterparts for values of $p \leq 1$. More specifically, we define

$$h^p_{at}(\partial\Omega) := F^{p,2}_0(\partial\Omega), \quad h^{1,p}_{at}(\partial\Omega) := F^{p,2}_1(\partial\Omega), \qquad \tfrac{n-1}{n} < p \leq 1. \tag{2.53}$$

In fact, introducing

$$h^p(\partial\Omega) := \begin{cases} h^p_{at}(\partial\Omega) \ \text{ if } \ \frac{n-1}{n} < p \le 1, \\ L^p(\partial\Omega) \ \text{ if } \ 1 < p < \infty, \end{cases} \qquad h^p_1(\partial\Omega) := \begin{cases} h^{1,p}_{at}(\partial\Omega) \ \text{ if } \ \frac{n-1}{n} < p \le 1, \\ L^p_1(\partial\Omega) \ \text{ if } \ 1 < p < \infty, \end{cases} \tag{2.54}$$

we can combine (2.51)-(2.53) into

$$h^p(\partial\Omega) = F^{p,2}_0(\partial\Omega), \quad h^p_1(\partial\Omega) = F^{p,2}_1(\partial\Omega), \qquad \tfrac{n-1}{n} < p < \infty. \tag{2.55}$$

The Hardy spaces thus defined turn out to coincide with those constructed based on atomic and molecular theories. Here we only briefly elaborate on this point.

The (inhomogeneous) Hardy space $h^p_{at}(\partial\Omega)$ has the following atomic characterization. Fix a threshold $r_o > 0$ and an index $1 < p_o \le \infty$. Call a function $a \in L^1(\partial\Omega)$ an inhomogeneous (p, p_o)-*atom* if for some surface ball $\Delta_r := B(X, r) \cap \partial\Omega$, $X \in \partial\Omega$, $r > 0$,

$$\begin{gathered} \operatorname{supp} a \subseteq \Delta_r, \quad \|a\|_{L^{p_o}(\partial\Omega)} \le r^{(n-1)\left(\frac{1}{p_o} - \frac{1}{p}\right)}, \text{ and} \\ \text{either } \ r = \eta, \quad \text{or } r < r_o \text{ and } \int_{\partial\Omega} a \, d\sigma = 0. \end{gathered} \tag{2.56}$$

Then $h^p_{at}(\partial\Omega)$ can be described as the ℓ^p-span of inhomogeneous (p, p_o)-atoms, with the infimum of the ℓ^p norm of the sequence of coefficients (taken over all representations) yielding an equivalent quasi-norm. This characterization allows to check that this is a “local” quasi-Banach space, in the sense that

$$h^p_{at}(\partial\Omega) \text{ is a module over } C^\alpha(\partial\Omega) \text{ for any } \alpha > (n-1)\left(\tfrac{1}{p} - 1\right). \tag{2.57}$$

The actual choices of the parameters p_o, r_o is immaterial, and

$$\left(h^p_{at}(\partial\Omega)\right)^* = C^{(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega). \tag{2.58}$$

Prior to discussing the dual of $h^1_{at}(\partial\Omega)$ we recall the local BMO space. For some fixed $0 < r_o < \operatorname{diam}(\partial\Omega)$, this is introduced as

$$f \in \operatorname{bmo}(\partial\Omega) \overset{def}{\Longleftrightarrow} f \in L^2(\partial\Omega) \quad \text{and} \quad \sup_{\substack{\Delta_r \text{ surface ball} \\ \text{with } r \le r_o}} \fint_{\Delta_r} |f - f_{\Delta_r}| \, d\sigma < \infty \tag{2.59}$$

(with $f_{\Delta_r} := \fint_{\Delta_r} f \, d\sigma$, where the barred integral indicates averaging), and is equipped with the natural norm. Then (cf. [9])

$$\Big(h^1_{at}(\partial\Omega)\Big)^* = \mathrm{bmo}\,(\partial\Omega) \quad \text{and} \quad h^1_{at}(\partial\Omega) = \Big(\mathrm{vmo}\,(\partial\Omega)\Big)^*, \tag{2.60}$$

where

$$f \in \mathrm{vmo}\,(\partial\Omega) \overset{def}{\Longleftrightarrow} f \in \mathrm{bmo}\,(\partial\Omega) \ \text{ and } \ \lim_{R\to 0}\left(\sup_{\substack{\Delta_r \text{ surface ball} \\ \text{with } r\le R}} \fint_{\Delta_r} |f - f_{\Delta_r}|\,d\sigma\right) = 0 \tag{2.61}$$

is Sarason's space of functions of vanishing mean oscillation. Let us point out that, for each $s \in (0,1)$, an alternative characterization of the latter space is

$$\mathrm{vmo}\,(\partial\Omega) = \text{the closure of } C^s(\partial\Omega) \text{ in } \mathrm{bmo}\,(\partial\Omega). \tag{2.62}$$

Later on, we shall also need the *homogeneous* Hardy space, which we denote by $H^p_{at}(\partial\Omega)$, defined for $\frac{n-1}{n} < p \le 1$ as follows:

$$H^p_{at}(\partial\Omega) := \Big\{ f = \sum_j \lambda_j a_j \,:\, a_j \ (p,p_o)\text{-atom},\ (\lambda_j)_j \in \ell^p \Big\}, \tag{2.63}$$

where the series converges in $\mathrm{Lip}(\partial\Omega)'$, the dual of the space of Lipschitz functions on $\partial\Omega$, and equipped with the usual infimum norm. Here, $1 < p_o \le \infty$ is a fixed parameter and a measurable function $a : \partial\Omega \to \mathbb{R}$ is called a (p,p_o)-*(homogeneous) atom* if there exists a surface ball $\Delta_r \subseteq \partial\Omega$ such that

$$\mathrm{supp}\, a \subseteq \Delta_r, \quad \|a\|_{L^{p_o}(\partial\Omega)} \le r^{-(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)} \quad \text{and} \quad \int_{\partial\Omega} a\,d\sigma = 0. \tag{2.64}$$

It is not too difficult to verify that

$$\begin{aligned} &h^p_{at}(\partial\Omega) = H^p_{at}(\partial\Omega) + \mathbb{R} = H^p_{at}(\partial\Omega) + L^q(\partial\Omega) \ \text{ for each } \ q > 1, \\ &h^p_{at}(\partial\Omega), \quad h^{1,p}_{at}(\partial\Omega) \ \text{ are modules over } \ \mathrm{Lip}\,(\partial\Omega). \end{aligned} \tag{2.65}$$

In relation to the regular Hardy space $h^{1,p}_{at}(\partial\Omega)$, we wish to mention that, if Ω, p, p_o are as before, and $r_o > 0$ is fixed, then

$$\begin{aligned} h^{1,p}_{at}(\partial\Omega) \ = \ &\Big\{ f \in \mathrm{Lip}\,(\partial\Omega)' : \ f = \sum_j \lambda_j a_j, \ (\lambda_j)_j \in \ell^p \text{ and } a_j \text{ regular } (p,p_o)\text{-atom} \\ &\qquad \text{supported in a surface ball of radius } \le r_o \text{ for every } j \Big\}, \end{aligned} \tag{2.66}$$

where the series converges in Lip $(\partial\Omega)'$, with $\|f\|_{h^{1,p}_{at}(\partial\Omega)}$ equivalent to the infimum of $\|(\lambda_j)_j\|_{\ell^p}$, taken over all representations $f = \sum_j \lambda_j a_j$. Here, if $(n-1)/n < p \le 1 < p_o \le \infty$, a function $a \in L^{p_o}_1(\partial\Omega)$ is called a *regular* (p,p_o)*-atom* if there exists a surface ball Δ_r so that

$$\operatorname{supp} a \subseteq \Delta_r, \qquad \|\nabla_{tan} a\|_{L^{p_o}(\partial\Omega)} \le r^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)}. \tag{2.67}$$

Also, if

$$\tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{n-1} \tag{2.68}$$

then

$$\begin{aligned} h^{1,p}_{at}(\partial\Omega) &= \Big\{ f \in L^{p^*}(\partial\Omega) : \partial_{\tau_{jk}} f \in H^p_{at}(\partial\Omega),\ 1 \le j,k \le n \Big\} \\ &= \Big\{ f \in L^{p^*}(\partial\Omega) : \partial_{\tau_{jk}} f \in h^p_{at}(\partial\Omega),\ 1 \le j,k \le n \Big\} \end{aligned} \tag{2.69}$$

and

$$\|f\|_{h^{1,p}_{at}(\partial\Omega)} \approx \|f\|_{L^{p^*}(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{H^p_{at}(\partial\Omega)} \approx \|f\|_{L^{p^*}(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{h^p(\partial\Omega)}. \tag{2.70}$$

We conclude this subsection with a brief discussion of smoothness spaces consisting of Whitney arrays. The general recipe is as follows. Given a smoothness space $\mathscr{X}$ on $\partial\Omega$, set

$$WA\big(\mathscr{X}\big) := \Big\{ \dot{f} = (f_0, f_1, ..., f_n) : \ f_j \in \mathscr{X},\ 0 \le j \le n, \ \text{and satisfy (2.21)} \Big\}, \tag{2.71}$$

which we equip with the (quasi-)norm

$$\|\dot{f}\|_{WA(\mathscr{X})} := \sum_{j=0}^{n} \|f_j\|_{\mathscr{X}}. \tag{2.72}$$

In this paper, we shall primarily work with $WA\big(B^{p,q}_s(\partial\Omega)\big)$ where s,p,q are as in (2.38), $WA\big(L^p(\partial\Omega)\big)$ and $WA\big(L^p_1(\partial\Omega)\big)$ with $1 < p < \infty$, as well as $WA\big(h^p_{at}(\partial\Omega)\big)$ and $WA\big(h^{1,p}_{at}(\partial\Omega)\big)$ considered for $\frac{n-1}{n} < p \le 1$.

2.4 The Kelvin transform associated to the bi-Laplacian

Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, be a set satisfying $0 \notin \Omega$ and assume that $u : \Omega \to \mathbb{R}$ is a function. Define the Kelvin transform of u as

$$\mathscr{K}[u](X) := |X|^{4-n} u\big(\tfrac{X}{|X|^2}\big), \qquad X \in \widetilde{\Omega}, \tag{2.73}$$

where

$$\widetilde{\Omega} := \Big\{X \in \mathbb{R}^n : \tfrac{X}{|X|^2} \in \Omega\Big\}. \tag{2.74}$$

Then $\mathscr{K}$ almost commutes with the bi-Laplacian in the sense that

$$\big(\Delta^2 \mathscr{K}[u]\big)(X) = \mathscr{K}[|X|^8 \Delta^2 u](X), \qquad X \in \widetilde{\Omega}, \tag{2.75}$$

if Ω open and $u \in C^4(\Omega)$. In particular, in this latter case,

$$\Delta^2 u \equiv 0 \;\text{ in }\; \Omega \implies \Delta^2 \mathscr{K}[u] \equiv 0 \;\text{ in }\; \widetilde{\Omega}. \tag{2.76}$$

It is straightforward to check that the Kelvin transform is linear, involutive, i.e.,

$$\mathscr{K}[\mathscr{K}[u]] = u, \tag{2.77}$$

and that, if $\alpha \in \mathbb{N}^n$ and $N \in \mathbb{R}$, then

$$\mathscr{K}\left[\tfrac{X^\alpha}{|X|^N} u\right](X) = \tfrac{X^\alpha}{|X|^{2|\alpha|-N}} \mathscr{K}[u](X). \tag{2.78}$$

We wish to study how other characteristics of the ambient change under the mapping

$$F : \Omega \longrightarrow \mathbb{R}^n, \quad F(X) := \tfrac{X}{|X|^2}. \tag{2.79}$$

Note that

$$F(\Omega) = \widetilde{\Omega} \quad \text{and} \quad F^{-1}(Y) = \tfrac{Y}{|Y|^2} \quad \text{for} \quad Y \in \widetilde{\Omega}. \tag{2.80}$$

In this scenario, the following result, of general nature, from [21] is going to be useful for us.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\mathcal{O}$ an open neighborhood of $\overline{\Omega}$, and let $F : \mathcal{O} \to \mathbb{R}^n$ be an orientation preserving C^∞-diffeomorphism.*

Then $\widetilde{\Omega} := F(\Omega)$ is a Lipschitz domain and if $\nu, \widetilde{\nu}$ and $\sigma, \widetilde{\sigma}$ are, respectively, the outward unit normals and surface measures on $\partial\Omega$ and $\partial\widetilde{\Omega}$, then

$$\widetilde{\nu} = \frac{(DF^{-1})^\top (\nu \circ F^{-1})}{|(DF^{-1})^\top (\nu \circ F^{-1})|}, \tag{2.81}$$

$$\widetilde{\sigma} = |(DF^{-1})^\top (\nu \circ F^{-1})| \, (|\det DF| \circ F^{-1}) \, F_* \sigma, \tag{2.82}$$

where $(DF^{-1})^\top$ *denotes the transposed of the Jacobian matrix of* F^{-1}*, and* $F_*\sigma$ *is the push-forward of the measure* σ*.*

Furthermore, for each $\alpha > 0$ *there exists* $\beta > 0$ *such that*

$$F(\Gamma_\alpha(X)) \subseteq \widetilde{\Gamma}_\beta(F(X)), \qquad X \in \partial\Omega, \tag{2.83}$$

where $\widetilde{\Gamma}_\beta$ *stands for the cone as in (2.1) relative to the domain* $\widetilde{\Omega}$*.*

Below, we study how Besov-Triebel-Lizorkin spaces change under mappings such as the one in (2.79).

Proposition 2.3. *In the context of Proposition 2.2, for every function* f *on* $\partial\Omega$

$$f \in B^{p,q}_s(\partial\Omega) \Longleftrightarrow f \circ F^{-1} \in B^{p,q}_s(\partial\widetilde{\Omega}), \tag{2.84}$$

if s, p, q *are as in (2.38), and*

$$f \in L^p(\partial\Omega) \Longleftrightarrow f \circ F^{-1} \in L^p(\partial\widetilde{\Omega}), \qquad f \in L^p_1(\partial\Omega) \Longleftrightarrow f \circ F^{-1} \in L^p_1(\partial\widetilde{\Omega}), \tag{2.85}$$

if $1 < p < \infty$*. Also, if* $\frac{n-1}{n} < p \leq 1$*,*

$$f \in h^{1,p}_{at}(\partial\Omega) \Longleftrightarrow f \circ F^{-1} \in h^{1,p}_{at}(\partial\widetilde{\Omega}), \tag{2.86}$$

and

$$f \in \mathrm{bmo}\,(\partial\Omega) \Longleftrightarrow f \circ F^{-1} \in \mathrm{bmo}\,(\partial\widetilde{\Omega}). \tag{2.87}$$

Moreover, if $0 < p < \infty$*, then for every function* u *in* Ω*,*

$$M(u) \in L^p(\partial\Omega) \Longleftrightarrow M(u \circ F^{-1}) \in L^p(\partial\widetilde{\Omega}). \tag{2.88}$$

Finally, anticipating notation introduced in § 4.2, for every distribution u *in* Ω

$$u \in B^{p,q}_\alpha(\Omega) \Longleftrightarrow u \circ F^{-1} \in B^{p,q}_\alpha(\widetilde{\Omega}), \tag{2.89}$$

$$u \in F^{p,q}_\alpha(\Omega) \Longleftrightarrow u \circ F^{-1} \in F^{p,q}_\alpha(\widetilde{\Omega}), \tag{2.90}$$

if $0 < p, q \leq \infty$ *(with* $p < \infty$ *in the case of Triebel-Lizorkin spaces), and* $\alpha \in \mathbb{R}$*.*

In all cases, a natural estimate holds.

Proof. When $p = q$, (2.84) follows from Proposition 2.2 and the equivalence

$$\|f\|_{B^{p,p}_s(\partial\Omega)} \approx \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(X)-f(Y)|^p}{|X-Y|^{n-1+sp}}\, d\sigma(X)d\sigma(Y)\right)^{1/p}, \tag{2.91}$$

valid for $(n-1)/n < p < \infty$ and $(n-1)(1/p-1)_+ < s < 1$, whenever $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with compact boundary. The case of off-diagonal Besov spaces is then a consequence of this and real interpolation.

Next, the first equivalence in (2.85) is a direct consequence of Proposition 2.2. Consider now the second equivalence in (2.85). For each $j,k \in \{1,\dots,n\}$, denoting by $\partial_{\widetilde{\tau}_{jk}}$ the tangential derivative on $\partial\widetilde{\Omega}$ given by $\widetilde{\nu}_j\partial_k - \widetilde{\nu}_k\partial_j$, we have

$$\begin{aligned}\partial_{\widetilde{\tau}_{jk}}(f\circ F^{-1}) &= \widetilde{\nu}_j\partial_k(f\circ F^{-1}) - \tilde{\nu}_k\partial_j(f\circ F^{-1})\\ &= \widetilde{\nu}_j\big((\partial_\ell f)\circ F^{-1}\big)\partial_k F^{-1}_\ell - \widetilde{\nu}_k\big((\partial_r f)\circ F^{-1}\big)\partial_j F^{-1}_r.\end{aligned} \tag{2.92}$$

Employing Proposition 2.2 we further write

$$\begin{aligned}\widetilde{\nu}_j\big((\partial_\ell f)\circ F^{-1}\big)\partial_k F^{-1}_\ell &= \big((DF^{-1})^\top(\nu\circ F^{-1})\big)_j\big(\nabla f\circ F^{-1}\big)_\ell(DF^{-1})_{\ell k}\\ &= \big[(DF^{-1})^\top\big((\nabla f\circ F^{-1})\otimes(\nu\circ F^{-1})\big)(DF^{-1})\big]_{kj},\end{aligned} \tag{2.93}$$

where for two vectors $a,b\in\mathbb{R}^n$ with $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$, we have set $a\otimes b$ to stand for the $n\times n$ matrix whose ij entry is given by

$$(a\otimes b)_{ij} := a_i b_j, \qquad i,j\in\{1,\dots,n\}. \tag{2.94}$$

Thus, based on (2.92) and (2.93),

$$\begin{aligned}\partial_{\widetilde{\tau}_{jk}}(f\circ F^{-1}) &= \big[(DF^{-1})^\top\big((\nabla f\circ F^{-1})\otimes(\nu\circ F^{-1})\big)(DF^{-1})\big]_{kj}\\ &\quad -\big[(DF^{-1})^\top\big((\nabla f\circ F^{-1})\otimes(\nu\circ F^{-1})\big)(DF^{-1})\big]_{jk}.\end{aligned} \tag{2.95}$$

This further gives,

$$\partial_{\widetilde{\tau}_{jk}}(f\circ F^{-1}) = \big[(DF^{-1})^\top\big(a\otimes b - b\otimes a\big)(DF^{-1})\big]_{kj} \tag{2.96}$$

where

$$a := \nabla f\circ F^{-1} \qquad \text{and} \qquad b := \nu\circ F^{-1}. \tag{2.97}$$

Since, generally speaking, $a\otimes b - b\otimes a = a_b\otimes b - b\otimes a_b$ where $a_b := a-(a\cdot b)b$, we may finally conclude that, for every j,k,

$$\partial_{\widetilde{\tau}_{jk}}(f\circ F^{-1})=\Big[(DF^{-1})^{\top}\big[\big(\nabla_{tan}f\otimes\nu-\nu\otimes\nabla_{tan}f\big)\circ F^{-1}\big](DF^{-1})\Big]_{kj}. \tag{2.98}$$

With this in hand, the second equivalence in (2.85) follows from (2.52) and Proposition 2.2. Note that (2.86) also follows from (2.98) and (2.66). In fact (2.87) is proved using similar arguments.

Moving on, (2.88) is implied by (2.83) and the well-known fact that, for every $\alpha,\beta>0$ and $0<p<\infty$,

$$\|M_\alpha(u)\|_{L^p(\partial\Omega)}\approx\|M_\beta(u)\|_{L^p(\partial\Omega)}, \tag{2.99}$$

uniformly in u. Finally, (2.89)-(2.90) follow from Rychkov's extension theorem in [49], and the fact that $B^{p,q}_\alpha(\mathbb{R}^n)$, $F^{p,q}_\alpha(\mathbb{R}^n)$ are locally invariant under C^∞-diffeomorphisms. This finishes the proof of the proposition. □

For the purpose of studying how the Dirichlet data for the bi-Laplacian changes under the transformation (2.73), we find it useful to introduce the following boundary Kelvin transform.

Definition 2.1. *Set*

$$\mathscr{K}_b(f_0,f_1,\cdots,f_n):=(g_0,g_1,\cdots,g_n), \tag{2.100}$$

where

$$g_0(Y):=|Y|^{4-n}f_0\big(\tfrac{Y}{|Y|^2}\big), \tag{2.101}$$

and, for each $k\in\{1,2,\cdots,n\}$,

$$g_k(Y):=(4-n)y_k|Y|^{2-n}f_0\big(\tfrac{Y}{|Y|^2}\big)+\big(\delta_{jk}-\tfrac{2y_jy_k}{|Y|^2}\big)|Y|^{2-n}f_j\big(\tfrac{Y}{|Y|^2}\big). \tag{2.102}$$

The above definition is designed to ensure that, at least formally,

$$\left.\begin{array}{l} f_0:=u\Big|_{\partial\Omega},\\ f_j:=(\partial_j u)\Big|_{\partial\Omega},\ 1\leq j\leq n,\\ v:=\mathscr{K}[u],\\ g_0:=v\Big|_{\partial\Omega},\\ g_j:=(\partial_j v)\Big|_{\partial\Omega},\ 1\leq j\leq n \end{array}\right\}\Longrightarrow(g_0,g_1,\cdots,g_n)=\mathscr{K}_b(f_0,f_1,\cdots,f_n). \tag{2.103}$$

Recall (2.71)-(2.72).

Proposition 2.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain for which $0 \notin \partial\Omega$, and define $\widetilde{\Omega}$ as in (2.74). Also, let s, p, q be as in (2.38). Then*

$$\mathscr{K}_b : WA\big(B_s^{p,q}(\partial\Omega)\big) \longrightarrow WA\big(B_s^{p,q}(\partial\widetilde{\Omega})\big) \tag{2.104}$$

is a well-defined, linear and bounded operator, and (with a slight abuse of notation)

$$\mathscr{K}_b\big(\mathscr{K}_b(f_0, f_1, \cdots, f_n)\big) = (f_0, f_1, \cdots, f_n) \tag{2.105}$$

for every $\dot{f} = (f_0, f_1, ..., f_n) \in WA\big(B_s^{p,q}(\partial\Omega)\big)$.

Moreover, similar properties hold for $\mathscr{K}_b$ considered on $WA\big(L^p(\partial\Omega)\big)$ and $WA\big(L_1^p(\partial\Omega)\big)$ if $1 < p < \infty$, as well as $WA\big(h_{at}^{1,p}(\partial\Omega)\big)$ if $\frac{n-1}{n} < p \leq 1$.

Proof. The fact that (2.104) is well-defined and bounded follows from (2.100)-(2.102), Proposition 2.3 and Proposition 2.2. The involutive property (2.105) is then a consequence of definitions and (2.77)-(2.78). The claims in the last part of the statement are proved analogously. □

Let $R > 0$ and assume that u is a bi-harmonic function outside the ball $B(0, R) \subset \mathbb{R}^n$, $n \geq 3$. Call u *biharmonic at infinity* if $\mathscr{K}[u]$, as a bi-harmonic function in $B(0, R^{-1}) \setminus \{0\}$, has a removable singularity at the origin. Generally speaking, it is known that a biharmonic function v in $B(0, R^{-1}) \setminus \{0\}$ has a removable singularity at 0 if

$$v(X) = o(|X|^{4-n}) \quad \text{as} \quad X \to 0. \tag{2.106}$$

In fact, continuing to assume that $n \geq 3$,

$$v \text{ has a removable singularity at } 0 \Longleftrightarrow \begin{cases} v(X) = o(|X|^{2-n}) \\ (\Delta v)(X) = o(|X|^{2-n}) \end{cases} \quad \text{as} \quad X \to 0. \tag{2.107}$$

See [22] for these and other related results.

Proposition 2.5. *Assume that u is a bi-harmonic function in the complement of a ball centered at the origin in $\mathbb{R}^n$, $n \geq 3$. Then u is bi-harmonic at infinity if and only if*

$$\lim_{X\to\infty} \frac{u(X)}{|X|^2} = 0 \quad \textit{and} \quad \lim_{X\to\infty} \Big\{2(4-n)u(X) - 4X \cdot (\nabla u)(X) + |X|^2(\Delta u)(X)\Big\} = 0. \tag{2.108}$$

Proof. By definition, u is biharmonic at infinity if and only if $v(X) := \mathscr{K}[u](X)$ has a removable singularity at 0. Employing (2.107) this further implies

$$u \text{ is biharmonic at infinity} \Longleftrightarrow \begin{cases} \lim\limits_{X\to 0} \frac{v(X)}{|X|^{2-n}} = \lim\limits_{X\to\infty} \frac{u(X)}{|X|^2} = 0, \\ \lim\limits_{X\to 0} \frac{(\Delta v)(X)}{|X|^{2-n}} = 0. \end{cases} \tag{2.109}$$

The last equality in the first line of the right-hand side of (2.109) justifies the first condition in (2.108). As for the second one, a straightforward calculation based on the definition of v shows that

$$\begin{aligned}\frac{\Delta v(X)}{|X|^{2-n}} &= 2n\,u\big(\tfrac{X}{|X|^2}\big)+\tfrac{1}{|X|^2}(\Delta u)\big(\tfrac{X}{|X|^2}\big)+2\frac{\nabla|X|^2\cdot\nabla\Big\{|X|^{2-n}u\big(\tfrac{X}{|X|^2}\big)\Big\}}{|X|^{2-n}}\\ &= 2(4-n)u\big(\tfrac{X}{|X|^2}\big)-4\tfrac{X}{|X|^2}\cdot(\nabla u)\big(\tfrac{X}{|X|^2}\big)+\tfrac{1}{|X|^2}(\Delta u)\big(\tfrac{X}{|X|^2}\big),\end{aligned} \tag{2.110}$$

from which the desired conclusion follows (after changing X into $\frac{X}{|X|^2}$). □

2.5 Singular integral operators related to the Laplacian

Recall that $\eta(X) := X$, for $X \in \mathbb{R}^n$, fix a bounded Lipschitz domain Ω in $\mathbb{R}^n$, and denote by Γ the canonical fundamental solution for the Laplacian $\Delta = \partial_j\partial_j$ in $\mathbb{R}^n$. That is,

$$\Gamma(X) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)}\dfrac{1}{|X|^{n-2}}, & \text{if } n \geq 3,\\ \dfrac{1}{2\pi}\log|X|, & \text{if } n = 2,\end{cases} \qquad X \in \mathbb{R}^n \setminus \{0\}, \tag{2.111}$$

where ω_n is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. With p.v. labeling boundary integrals which are taken in the principal value sense, then set

$$Rf(X) := \text{p.v.}\int_{\partial\Omega}\langle X, (\nabla\Gamma)(X-Y)\rangle f(Y)\,d\sigma(Y), \quad X \in \partial\Omega \tag{2.112}$$

$$\mathcal{R}f(X) := \int_{\partial\Omega}\langle Y, (\nabla\Gamma)(X-Y)\rangle f(Y)\,d\sigma(Y), \quad X \notin \partial\Omega. \tag{2.113}$$

Also, recall the harmonic single layer and its boundary version given, respectively, by

$$\mathcal{S}f(X) := \int_{\partial\Omega}\Gamma(X-Y)f(Y)\,d\sigma(Y), \quad X \notin \partial\Omega, \tag{2.114}$$

$$Sf(X) := \int_{\partial\Omega}\Gamma(X-Y)f(Y)\,d\sigma(Y), \quad X \in \partial\Omega. \tag{2.115}$$

Recall (2.1)-(2.3).

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an arbitrary, bounded Lipschitz domain. Then the following hold:*

(1) $\langle X, \nabla\mathcal{S}f(X)\rangle\Big|_{\partial\Omega_\pm} = \big(\mp\tfrac{1}{2}\langle\eta,\nu\rangle I + R\big)\,f(X)$, $X \in \partial\Omega$.

(2) $R^*f(X) = -\text{p.v.} \int_{\partial\Omega} \langle Y, (\nabla\Gamma)(X-Y)\rangle f(Y)\, d\sigma(Y)$, $X \in \partial\Omega$.

(3) $\mathcal{R}f\Big|_{\partial\Omega_\pm}(X) = -\left(\pm\frac{1}{2}\langle\eta,\nu\rangle I + R^*\right) f(X)$, $X \in \partial\Omega$.

(4) $\mathcal{R}f(X) = div\,\mathcal{S}(f\eta)(X)$, $X \notin \partial\Omega$.

(5) $\|M(\mathcal{R}f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$, $1 < p < \infty$.

(6) R, $R^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$, *boundedly for* $1 < p < \infty$.

(7) $Rf + R^*f = (2-n)Sf$ *on* $\partial\Omega$ *if* $n \ge 3$, *and* $Rf + R^*f = -\frac{1}{2\pi}\int_{\partial\Omega} f\, d\sigma$ *on* $\partial\Omega$ *if* $n = 2$.

(8) $\nabla_\eta\mathcal{S}f - div\,\mathcal{S}(f\eta) = (2-n)\mathcal{S}f$ *if* $n \ge 3$, *and* $\nabla_\eta\mathcal{S}f - div\,\mathcal{S}(f\eta) = -\frac{1}{2\pi}\int_{\partial\Omega} f\, d\sigma$ *if* $n = 2$, *in* $\mathbb{R}^n \setminus \partial\Omega$.

Proof. Properties (1)-(6) follow from the fact that

$$\|M(\nabla\mathcal{S}f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)} \tag{2.116}$$

and, with $\partial_j S$ denoting the boundary integral operator with kernel $(\partial_j\Gamma)(X-Y)$,

$$\partial_j\mathcal{S}f\Big|_{\partial\Omega_\pm} = \left(\mp\frac{1}{2}\nu_j I + \text{p.v.}\partial_j S\right) f, \quad 1 \le j \le n. \tag{2.117}$$

As for (7), observe from (2.111) that $\nabla_\eta\Gamma = (2-n)\Gamma$ if $n \ge 3$, and $\nabla_\eta\Gamma = -\frac{1}{2\pi}$ if $n = 2$. Since the integral kernel of $R + R^*$ is $(\nabla_\eta\Gamma)(X-Y)$, the desired conclusion follows. The identity in (8) is proved similarly. □

Recall next the so-called harmonic double layer

$$\mathcal{D}f(X) := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{\langle Y - X, \nu(Y)\rangle}{|X-Y|^n} f(Y)\, d\sigma(Y), \quad X \notin \partial\Omega, \tag{2.118}$$

and, with $\partial_\nu = \nu\cdot\nabla$ denoting the normal derivative, observe that

$$\mathcal{D}f(X) = \int_{\partial\Omega} \partial_{\nu(Y)}\Big[\Gamma(X-Y)\Big] f(Y)\, d\sigma(Y). \tag{2.119}$$

Consequently, for every index $j \in \{1, \dots, n\}$ and every function $f \in L^p_1(\partial\Omega)$, $1 < p < \infty$, we have

$$\partial_j\mathcal{D}f(X) = \partial_k\mathcal{S}(\partial_{\tau_{jk}}f)(X). \tag{2.120}$$

In particular, using

$$\nu_j \nu_k \partial_{\tau_{jk}} = 0 \tag{2.121}$$

we obtain

$$\partial_\nu \mathcal{D} f = \tfrac{1}{2} \partial_{\tau_{jk}} S(\partial_{\tau_{jk}} f). \tag{2.122}$$

Let us also record here the jump-formulas for the harmonic double layer, namely,

$$\mathcal{D} f \Big|_{\partial\Omega_\pm} = \big(\pm\tfrac{1}{2} I + K\big) f \tag{2.123}$$

where

$$K f(X) := \frac{1}{\omega_{n-1}} \text{p.v.} \int_{\partial\Omega} \frac{\langle Y - X, \nu(Y)\rangle}{|X - Y|^n} f(Y)\, d\sigma(Y), \quad X \in \partial\Omega. \tag{2.124}$$

Then $\partial_\nu^\pm \mathcal{S} f = (\mp\frac{1}{2} I + K^*) f$ where K^* is the adjoint of K, and $\partial_\nu^\pm$ indicates that the normal derivative has been taken by approaching the boundary from inside $\Omega_\pm$. Also, given (2.123), it follows that the so-called "Poisson integral" operator, mapping a boundary function into its harmonic extension in the given domain, can then be expressed as

$$\mathscr{P} := \mathcal{D}\Big(\tfrac{1}{2} I + K\Big)^{-1}, \tag{2.125}$$

granted that the inverse exists.

Lemma 2.7. *If $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded, Lipschitz domain which is star-like with respect to the origin, then there exists $\varepsilon > 0$ such that the mappings*

$$\begin{aligned} &\tfrac{1}{2}\langle \eta, \nu\rangle I + R : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \\ &\tfrac{1}{2}\langle \eta, \nu\rangle I + R^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \end{aligned} \tag{2.126}$$

are isomorphisms for every $p \in (2 - \varepsilon, 2 + \varepsilon)$.

Proof. By stability and duality it is enough to show that the mapping

$$\tfrac{1}{2}\langle \eta, \nu\rangle I + R : L^2(\partial\Omega) \to L^2(\partial\Omega) \tag{2.127}$$

is invertible. To this end, write

$$\tfrac{1}{2}\langle \eta, \nu\rangle I + R = \Big[\tfrac{1}{2}\langle \eta, \nu\rangle I + \Big(\frac{R - R^*}{2}\Big)\Big] + \Big(\frac{R + R^*}{2}\Big) \tag{2.128}$$

and observe that $R + R^*$ has integral kernel $\langle X - Y, (\nabla\Gamma)(X - Y)\rangle = O(|X - Y|^{2-n})$. Thus, $R + R^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is compact for all $p \in (1, \infty)$. On the other hand, since $\langle \eta, \nu\rangle \geq \kappa > 0$ almost everywhere on $\partial\Omega$, it follows that $\frac{1}{2}\langle \eta, \nu\rangle I + \big(\frac{R-R^*}{2}\big)$ is accretive on $L^2(\partial\Omega)$, hence invertible on $L^2(\partial\Omega)$. All together, this proves that the operator in (2.127) is Fredholm with index zero.

There remains to show that it is also one-to-one. To see this, let $f \in L^2(\partial\Omega)$ be such that $\big(\frac{1}{2}\langle \eta, \nu\rangle I + R\big) f = 0$ and set $u := \nabla_\eta \mathcal{S} f$ in $\mathbb{R}^n \setminus \bar{\Omega}$. Then by (2.32), (2.117) and Lemma 2.6,

$$\begin{cases} \Delta u = 0 \ \text{ in } \ \mathbb{R}^n \setminus \bar{\Omega}, \\ M(u) \in L^2(\partial\Omega), \\ u(X) = O(|X|^{2-n}) \ \text{ as } \ |X| \to \infty, \\ u\Big|_{\partial\Omega} = 0. \end{cases} \tag{2.129}$$

Uniqueness in the exterior Dirichlet boundary value problem then gives that $\nabla_\eta \mathcal{S} f = 0$ in $\mathbb{R}^n \setminus \bar{\Omega}$. Fix $r > 1$ and $X \in \mathbb{R}^n \setminus \bar{\Omega}$ and write

$$0 = \int_1^r (\nabla_\eta \mathcal{S} f)(tX)\, \frac{dt}{t} = (\mathcal{S} f)(rX) - (\mathcal{S} f)(X). \tag{2.130}$$

We claim that this forces $\mathcal{S} f = 0$ in $\mathbb{R}^n \setminus \bar{\Omega}$. Accepting this for a moment, note that going to the boundary yields $S f = 0$ on $\partial\Omega$, hence $\mathcal{S} f = 0$ in Ω, by the uniqueness in the L^2-Dirichlet problem in Ω. Thus, ultimately, $f = \partial_\nu^- \mathcal{S} f - \partial_\nu^+ \mathcal{S} f = 0$. This proves that the operator in (2.127) is also one-to-one, thus, invertible.

Returning to the claim made above, when $n \geq 3$ let $r \to \infty$ in (2.130) and obtain $(\mathcal{S} f)(rX) \to 0$ so ultimately $\mathcal{S} f = 0$ in $\mathbb{R}^n \setminus \bar{\Omega}$, as wanted. When $n = 2$, (2.130) gives $\mathcal{S} f(X) = \frac{1}{2\pi} \int_{\partial\Omega} [\log r + \log |X - (Y/r)|] f(Y)\, d\sigma(Y)$. Analyzing two cases, $\int_{\partial\Omega} f\, d\sigma \neq 0$ and $\int_{\partial\Omega} f\, d\sigma = 0$, and then passing $r \to \infty$, we once again arrive at the conclusion that $\mathcal{S} f = 0$ in $\mathbb{R}^n \setminus \bar{\Omega}$. This justifies the claim and finishes the proof of the lemma. □

The lemma below is going to be useful in § 3.5.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded Lipschitz domain. Also, assume that $m \in L^\infty(\partial\Omega)$ is such that there exists $\kappa > 0$ for which $m(X) \geq \kappa$ for a.e. $X \in \partial\Omega$. Then*

$$I - m\, S : h^p(\partial\Omega) \longrightarrow h^p(\partial\Omega) \tag{2.131}$$

is an invertible operator for $\frac{n-1}{n} < p < \infty$.

Proof. Since the operator in question is Fredholm with index zero, it suffices to show that $I - m\, S$ is one-to-one on $L^2(\partial\Omega)$. To this end, we observe that for every $f \in L^2(\partial\Omega)$

$$\int_{\partial\Omega} (\mp\tfrac{1}{2} I + K^*) f\, S f\, d\sigma = \pm \int_{\Omega_\pm} |\nabla \mathcal{S} f|^2\, dX. \tag{2.132}$$

Subtracting the two versions of (2.132) then gives

$$\int_{\partial\Omega} f\, Sf\, d\sigma = -\int_{\mathbb{R}^n} |\nabla \mathcal{S} f|^2\, dX \leq 0. \tag{2.133}$$

Hence $S \leq 0$ on $L^2(\partial\Omega)$ which, in turn, implies that $I - m\, S$ is one-to-one on $L^2(\partial\Omega)$. □

We continue to review material which is going to be useful in § 3.5. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, let f_o be the unique function such that

$$f_o \in L^{2+\varepsilon}(\partial\Omega), \qquad \int_{\partial\Omega} f_o\, d\sigma \neq 0, \quad \text{and} \quad Sf_o \equiv 1 \;\text{ on }\; \partial\Omega, \tag{2.134}$$

where $\varepsilon = \varepsilon(\Omega) > 0$. Then

$$S : \Big\{ f \in L^p(\partial\Omega) : \int_{\partial\Omega} f f_o\, d\sigma = 0 \Big\} \longrightarrow L^p_1(\partial\Omega)/\mathbb{R} \tag{2.135}$$

is an isomorphism if $1 < p < 2 + \varepsilon$.

Although we will not need this here, let us nonetheless mention that

$$-\tfrac{1}{2}\langle \eta, \nu\rangle I + R : \Big\{ f \in L^p(\partial\Omega) : \int_{\partial\Omega} f f_o\, d\sigma = 0 \Big\} \longrightarrow L^p_\omega(\partial\Omega) \tag{2.136}$$

is an isomorphism whenever $2 - \varepsilon < p < 2 + \varepsilon$. Above,

$$L^p_\omega(\partial\Omega) := \{ f \in L^p(\partial\Omega) : \mathscr{P} f(0) = 0 \}. \tag{2.137}$$

In relation to this, we make the following observation which is going to be useful for us later.

Lemma 2.9. *If Ω is as in (2.4), then*

$$|\eta|^2 \notin L^p_\omega(\partial\Omega), \qquad \forall\, p > 2 - \varepsilon. \tag{2.138}$$

Proof. Let $u := \mathscr{P}|\eta|^2$. Then u is harmonic in Ω and continuous on $\bar\Omega$. The Maximum Principle then gives

$$u(0) \geq \min_{\bar\Omega} u = \min_{\partial\Omega} u = \min_{X \in \partial\Omega} |X|^2 = \mathrm{dist}\,(0, \partial\Omega)^2 > 0. \tag{2.139}$$

Hence, $u(0) \neq 0$, proving (2.138). □

We now briefly recall the Newtonian volume potential for the Laplacian. Specifically, given a function $v \in L^1(\Omega)$, we set

$$(\Gamma v)(X) := \int_\Omega \Gamma(X-Y)v(Y)\,dY, \qquad X \in \mathbb{R}^n, \tag{2.140}$$

and note that

$$\Gamma v \text{ is harmonic in } \Omega_-, \tag{2.141}$$

and

$$(\partial^\alpha \Gamma v)\Big|_{\partial\Omega_+} = (\partial^\alpha \Gamma v)\Big|_{\partial\Omega_-}, \qquad \text{if } \ |\alpha| \le 1. \tag{2.142}$$

Also, if we denote by B the canonical fundamental solution for Δ^2 in $\mathbb{R}^n$, i.e.,

$$B(X) := \begin{cases} \dfrac{|X|^{4-n}}{2(n-4)(n-2)\omega_n} & \text{if } n=3 \text{ or } n>4, \\ -\dfrac{1}{4\omega_4}\log|X| & \text{if } n=4, \\ \dfrac{1}{8\pi}|X|^2\log|X| & \text{if } n=2, \end{cases} \qquad X \in \mathbb{R}^n \setminus \{0\}, \tag{2.143}$$

then

$$\Gamma(X) = \Delta B(X), \qquad X \in \mathbb{R}^n \setminus \{0\}. \tag{2.144}$$

Throughout the paper, $\nabla\nabla$ will denote generic combinations of second order partial derivatives.

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then for every $p \in (\frac{n-1}{n}, \infty)$ there exists $C = C(\Omega, p) > 0$ such that*

$$\|M(\nabla\nabla\Gamma(\nabla_\eta \mathcal{S} f))\|_{L^p(\partial\Omega)} \le C\|f\|_{h^p(\partial\Omega)}. \tag{2.145}$$

Proof. We shall show that

$$\begin{aligned} \int_\Omega \Gamma(X-Y)\nabla_\eta \mathcal{S} f(Y)\,dY \ &= \ \int_{\partial\Omega} \nabla_\eta^Y\Big[B(X-Y)\Big](-\tfrac{1}{2}I + K^*)f(Y)\,d\sigma(Y) \\ &\quad - \int_{\partial\Omega} y_j\,\partial_k^Y\Big[B(X-Y)\Big](\partial_{\tau_{jk}} Sf)(Y)\,d\sigma(Y) \\ &\quad + (n-2)\int_{\partial\Omega} B(X-Y)(-\tfrac{1}{2}I + K^*)f(Y)\,d\sigma(Y). \end{aligned} \tag{2.146}$$

From this and Calderón-Zygmund theory, (2.145) follows.

To this end, let us first make the observation that for any two reasonable functions F, G and any $k \in \{1, ..., n\}$, there holds

$$\int_\Omega \partial_k F \nabla_\eta G \, dX = \int_\Omega \nabla_\eta F \partial_k G \, dX - \int_{\partial\Omega} F \eta_j \partial_{\tau_{jk}} G \, d\sigma + (n-1) \int_\Omega F \partial_k G \, dX. \quad (2.147)$$

For example, (2.147) is valid if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and

$$\begin{gathered} F, G \in C^1(\Omega), \quad |\nabla F||\nabla G| \in L^1(\Omega), \quad M(|F||\nabla G|) \in L^p(\partial\Omega) \text{ some } p > 1 \\ \text{and } \; F\nabla G \text{ has a nontangential trace on } \partial\Omega. \end{gathered} \quad (2.148)$$

Indeed, under the above assumptions, the Divergence Theorem applied to the vector field $U := -F\partial_k G \eta + F \nabla_\eta G e_k$ yields (2.147). Using this identity we have

$$\begin{aligned} \int_\Omega \Gamma(X-Y) \nabla_\eta \mathcal{S} f(Y) \, dY &= \int_\Omega \partial_k^Y \partial_k^Y \Big[B(X-Y) \Big] \nabla_\eta \mathcal{S} f(Y) \, dY \qquad (2.149) \\ &= \int_\Omega \nabla_\eta^Y \partial_k^Y \Big[B(X-Y) \Big] \partial_k \mathcal{S} f(Y) \, dY \\ &\quad - \int_{\partial\Omega} y_j \, \partial_k^Y \Big[B(X-Y) \Big] (\partial_{\tau_{jk}} S f)(Y) \, d\sigma(Y) \\ &\quad + (n-1) \int_\Omega \partial_k^Y \Big[B(X-Y) \Big] \partial_k \mathcal{S} f(Y) \, dY. \end{aligned}$$

For the first solid integral in the right-most side above use (2.29) then integrate ∂_k^Y by parts and use the fact that $\partial_k \partial_k \mathcal{S} f = 0$ and $\nu_k \partial_k \mathcal{S} f = (-\frac{1}{2} I + K^*) f$ on $\partial\Omega$. We also integrate by parts in the last solid integral in order to finally obtain (2.146). □

In relation to the estimate proved in Lemma 2.10 we make the following:

Remark. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ and $C > 0$ with the following significance. For any harmonic function v in Ω such that $M(v) \in L^p(\partial\Omega)$ for some $p > 2 - \varepsilon$, we have $M(\nabla\nabla\Gamma(v)) \in L^p(\partial\Omega)$ and

$$\|M(\nabla\nabla\Gamma(v))\|_{L^p(\partial\Omega)} \leq C \|v\|_{L^p(\partial\Omega)}. \quad (2.150)$$

To see this, note that any function v which is harmonic in Ω and satisfies $M(v) \in L^p(\partial\Omega)$ for some $p > 2 - \varepsilon$ (with $\varepsilon > 0$ small enough, depending on Ω), can be represented in the form $v = \mathcal{D} f$ for some $f \in L^p(\partial\Omega)$. Consequently, $\Gamma(v) = \Gamma(\mathcal{D} f)$ so that the membership $M(\nabla\nabla\Gamma(v)) \in L^p(\partial\Omega)$ and the estimate (2.150) follow immediately from (2.189) and Calderón-Zygmund theory.

We now record an estimate, modeled upon lemma on page 213 of [53]. Given the usefulness of this type of result in a variety of situations, below we state and prove a slightly more general version than the one we shall actually need later on.

Lemma 2.11. *Suppose $\Gamma^a_\alpha \subset \Gamma^b_\beta \subset \mathbb{R}^n$ are two coaxial, truncated, circular cones with a common vertex $X^* \in \mathbb{R}^n$, whose apertures α, β and heights a, b satisfy $0 < \alpha < \beta < \pi$ and $0 < a < b < \infty$. Also, fix a vector $\xi \in S^{n-1} \cap \{t(X - X^*) : X \in \Gamma^a_\alpha,\, t > 0\}$ and assume that $u \in C^\infty(\Gamma^b_\beta)$ is a null-solution of a homogeneous, constant coefficient elliptic operator L. Then, for each $1 \leq q < \infty$ and $-\infty < r < n-1$, there exist a constant $C > 0$ and a small compact neighborhood $\mathcal{O}$ of the top lid of Γ^a_α, both depending only on $a, b, \alpha, \beta, r, L, \xi, q, n$, such that*

$$\int_{\Gamma^a_\alpha} \frac{|\nabla u(X)|^q}{|X - X^*|^r}\, dX \leq C \int_{\Gamma^b_\beta} \frac{|\nabla_\xi u(X)|^q}{|X - X^*|^r}\, dX + C \sup_{X \in \mathcal{O}} |u(X)|^q, \tag{2.151}$$

where ∇_ξ is the directional derivative along ξ.

Proof. Performing a rotation and a translation, there is no loss of generality in assuming that $X^* = 0$ and $\xi = e_n$. In particular, $\nabla_\xi = \partial_n$. First, using the Fundamental Theorem of Calculus and interior estimates, we observe that for each point $X = (x', x_n) \in \Gamma^a_\alpha$ and each fixed index $j \in \{1, ..., n\}$,

$$\begin{aligned} |(\partial_j u)(X)| &= \Big|(\partial_j u)(x', a) - \int_{x_n}^a (\partial_n \partial_j u)(x', t)\, dt\Big| \\ &\leq \int_{x_n}^a |(\partial_j \partial_n u)(x', t)|\, dt + C \sup_{\mathcal{O}} |u|, \end{aligned} \tag{2.152}$$

for some small compact neighborhood $\mathcal{O}$ of the top lid of Γ^a_α. Using once again interior estimates, we have for $x_n < t < a$,

$$|(\partial_j \partial_n u)(x', t)| \leq \frac{C}{t} \Big(⨍_{B_{\lambda t}(x', t)} |(\partial_n u)(Y)|^q\, dY \Big)^{\frac{1}{q}}, \tag{2.153}$$

where $\lambda \in (0, 1)$ is chosen to be small enough that $B_{\lambda t}(x', t) \subset \Gamma^b_\beta$ if $(x', t) \in \Gamma^a_\alpha$. Define

$$D_t := \{Y = (y', y_n) \in \Gamma^b_\beta : |y_n - t| < \lambda t\}, \qquad t > 0. \tag{2.154}$$

Since $B_{\lambda t}(x', t) \subset D_t$ for every $t \in (0, a)$ and every $X = (x', x_n) \in \Gamma^a_\alpha$, it follows that

$$|(\partial_j \partial_n u)(x', t)| \leq C\, t^{-(n+q)/q} F(t)^{1/q}, \qquad \forall\, t \in (0, a), \tag{2.155}$$

if $X = (x', x_n) \in \Gamma^a_\alpha$, where we have set

$$F(t) := \int_{D_t} |(\partial_n u)(Y)|^q\, dY, \qquad 0 < t < a. \tag{2.156}$$

Consider the spherical cap $S_\alpha := S^{n-1} \cap \{tX : X \in \Gamma^a_\alpha, t > 0\}$. Then, combining (2.152) and (2.155) and taking into account the fact that $(s\omega)_n = s\omega_n \geq cs$ for some $c = c(\alpha) \in (0, 1)$, we obtain from (2.152) and (2.155) that

$$|(\partial_j u)(s\omega)| \leq C \int_{cs}^{a} t^{-(n+q)/q} F(t)^{1/q}\, dt + C \sup_{\mathcal{O}} |u|, \tag{2.157}$$

uniformly for $\omega \in S_\alpha$ and $0 < s < a$. Applying Hardy's inequality, we thus obtain

$$\int_0^a |(\partial_j u)(s\omega)|^q\, s^{n-1-r}\, ds \leq C \int_0^a t^{-1-r} F(t)\, dt + C \sup_{\mathcal{O}} |u|^q, \tag{2.158}$$

uniformly for $\omega \in S_\alpha$. Let χ_t denote the characteristic function of D_t in Γ^b_β. Then by Fubini's Theorem,

$$\begin{aligned} \int_0^a t^{-1-r} F(t)\, dt &= \int_0^a t^{-1-r} \Big(\int_{D_t} |(\partial_n u)(Y)|^q\, dY \Big)\, dt \\ &= \int_{\Gamma^b_\beta} |(\partial_n u)(Y)|^q \Big(\int_0^a t^{-1-r} \chi_t(Y)\, dt \Big)\, dY \\ &\leq \int_{\Gamma^b_\beta} |(\partial_n u)(Y)|^q \Big(\int_{y_n/(1+\lambda)}^{y_n/(1-\lambda)} t^{-1-r}\, dt \Big)\, dY \\ &\leq C \int_{\Gamma^b_\beta} |(\partial_n u)(Y)|^q\, y_n^{-r}\, dY. \end{aligned} \tag{2.159}$$

Then using (2.158) and (2.159) and polar coordinates, we have

$$\begin{aligned} \int_{\Gamma^a_\alpha} |(\partial_j u)(X)|^q\, x_n^{-r}\, dX &\leq \int_{S_\alpha} \int_0^a |(\partial_j u)(s\omega)|^q\, s^{n-1-r}\, ds\, d\omega \\ &\leq C \int_{\Gamma^b_\beta} |(\partial_n u)(X)|^q\, x_n^{-r}\, dX + C \sup_{\mathcal{O}} |u|^q. \end{aligned} \tag{2.160}$$

Since $j \in \{1, ..., n\}$ was arbitrary, this finishes the proof of the lemma. □

Lemma 2.12. *Let Ω be as in (2.4) and assume that v is harmonic in Ω. Then for every $p \in (0, \infty)$ there exists a finite constant $C = C(\Omega, p) > 0$ such that*

$$\|M(\nabla v)\|_{L^p(\partial\Omega)} \leq C \|M(\nabla_\eta v)\|_{L^p(\partial\Omega)}. \tag{2.161}$$

Proof. There are two basic ingredients used in the proof. One is E. Stein's Lemma 2.11. The second one is B. Dahlberg's equivalence

$$\|M_\alpha(w)\|_{L^p(\partial\Omega)} \approx \|\mathcal{A}_\alpha(w)\|_{L^p(\partial\Omega)}, \tag{2.162}$$

valid for every $0 < p < \infty$, uniformly for functions satisfying $\Delta w = 0$ in Ω and normalized such that $w(0) = 0$. Recall that the area-function $\mathcal{A}$ is given by

$$(\mathcal{A}_\alpha w)(X) := \left(\int_{\Gamma_\alpha(X)} \frac{|\nabla w(Y)|^2}{|X-Y|^{n-2}}\,dY\right)^{1/2}, \qquad X \in \partial\Omega, \tag{2.163}$$

where, given $\alpha > 0$, $\Gamma_\alpha(X)$ is the nontangential approach region with vertex at $X \in \partial\Omega$ (cf. (2.1)). For a suitable relatively compact subset $\mathcal{O}$ of Ω, we now estimate

$$\begin{aligned} \mathcal{A}_\alpha(\nabla v)(X)^2 &= \int_{\Gamma_\alpha(X)} \frac{|\nabla\nabla v(Y)|^2}{|X-Y|^{n-2}}\,dY \leq C\int_{\Gamma_\beta(X)} \frac{|\nabla_X\nabla v(Y)|^2}{|X-Y|^{n-2}}\,dY + C\sup_{\mathcal{O}}|\nabla v|^2 \\ &\leq C\int_{\Gamma_\beta(X)} \frac{|\nabla_\eta\nabla v(Y)|^2}{|X-Y|^{n-2}}\,dY + C\sup_{\mathcal{O}}|\nabla v|^2 \\ &\quad + C\int_{\Gamma_\beta(X)} \frac{|X-Y|^2|\nabla\nabla v(Y)|^2}{|X-Y|^{n-2}}\,dY. \end{aligned} \tag{2.164}$$

The first inequality above is a consequence of Lemma 2.11 (with $q = 2$ and $r = n-2$), while the second inequality follows by using the generic estimate

$$|\nabla_X w(Y)| = |\langle X, (\nabla w)(Y)\rangle| \leq |(\nabla_\eta w)(Y)| + |X-Y||(\nabla w)(Y)| \tag{2.165}$$

with $w := \nabla v$. Going further, for a suitably small $\varepsilon > 0$, and $\alpha > 0$, $X \in \partial\Omega$, define

$$\Gamma_\alpha^\varepsilon(X) := \Big\{Y \in \Omega :\, |X-Y| \leq \min\{\varepsilon\,,\,(1+\alpha)\,\mathrm{dist}\,(Y,\partial\Omega)\}\Big\}, \tag{2.166}$$

i.e., the nontangential approach region with vertex at $X \in \partial\Omega$, truncated at height ε. We then majorize the last term in (2.164) by

$$\begin{aligned} C\varepsilon^2\int_{\Gamma_\beta^\varepsilon(X)} \frac{|\nabla\nabla v(Y)|^2}{|X-Y|^{n-2}}\,dY + C_\varepsilon\int_{\Gamma_\beta(X)\setminus\Gamma_\beta^\varepsilon(X)} |\nabla\nabla v(Y)|^2\,dY \\ \leq C\varepsilon^2\mathcal{A}_\beta(\nabla v)(X)^2 + C_\varepsilon\sup_{\mathcal{O}_\varepsilon}|v|^2, \end{aligned} \tag{2.167}$$

by interior estimates, where $\mathcal{O}_\varepsilon$ is a certain relatively compact subset of Ω. Using this and (2.29), we obtain from (2.164) that, for every $X \in \partial\Omega$,

$$
\begin{aligned}
\mathcal{A}_\alpha(\nabla v)(X) &\leq C\mathcal{A}_\beta(\nabla_\eta v)(X) + C\varepsilon\mathcal{A}_\beta(\nabla v)(X) \\
&\quad + C\mathcal{A}_\beta(v)(X) + C_\varepsilon \sup_{\mathcal{O}_\varepsilon} |v|,
\end{aligned} \tag{2.168}
$$

where C is independent of ε.

After this preamble, we invoke (2.162) and (2.168) in order to write

$$
\begin{aligned}
\|M_\alpha(\nabla v)\|_{L^p(\partial\Omega)} &\leq C\|\mathcal{A}_\alpha(\nabla v)\|_{L^p(\partial\Omega)} + C|\nabla v(0)| \\
&\leq C\|\mathcal{A}_\beta(\nabla_\eta v)\|_{L^p(\partial\Omega)} + C\varepsilon\|\mathcal{A}_\beta(\nabla v)\|_{L^p(\partial\Omega)} \\
&\quad + C\|\mathcal{A}_\beta(v)\|_{L^p(\partial\Omega)} + C_\varepsilon \sup_{\mathcal{O}_\varepsilon} |v|.
\end{aligned} \tag{2.169}
$$

By relying once more on (2.162) and recalling (2.99), the right-most expression above is dominated by

$$
\begin{aligned}
& C\|M_\beta(\nabla_\eta v)\|_{L^p(\partial\Omega)} + C\varepsilon\|M_\beta(\nabla v)\|_{L^p(\partial\Omega)} + C\|M_\beta(v)\|_{L^p(\partial\Omega)} + C_\varepsilon \sup_{\mathcal{O}_\varepsilon} |v| \\
&\qquad \leq C\|M_\alpha(\nabla_\eta v)\|_{L^p(\partial\Omega)} + C\varepsilon\|M_\alpha(\nabla v)\|_{L^p(\partial\Omega)} + C_\varepsilon\|M_\alpha(v)\|_{L^p(\partial\Omega)}.
\end{aligned} \tag{2.170}
$$

Utilizing this back in (2.169) and choosing $\varepsilon > 0$ sufficiently small so that the term with coefficient $C\varepsilon$ can be absorbed in the left-hand side, we obtain

$$
\|M_\alpha(\nabla v)\|_{L^p(\partial\Omega)} \leq C\|M_\alpha(\nabla_\eta v)\|_{L^p(\partial\Omega)} + C\|M_\alpha(v)\|_{L^p(\partial\Omega)}. \tag{2.171}
$$

At this stage, there remains to eliminate the last term in (2.171). With this goal in mind, and given the nature of the inequality (2.161), we may (and will) assume that $v(0) = 0$ to begin with. Then for $X \in \partial\Omega$ and $Y \in \Gamma_\alpha(X)$,

$$
v(Y) = \int_0^1 (\nabla_\eta v)(tY)\,\frac{dt}{t} = \int_0^\varepsilon (\nabla_\eta v)(tY)\,\frac{dt}{t} + \int_\varepsilon^1 (\nabla_\eta v)(tY)\,\frac{dt}{t}, \tag{2.172}
$$

where $\varepsilon > 0$ is a small number, to be specified shortly. Hence,

$$
\begin{aligned}
|v(Y)| &\leq C\int_0^\varepsilon |(\nabla v)(tY)|\,dt + C\varepsilon^{-1}\int_\varepsilon^1 |(\nabla_\eta v)(tY)|\,dt \\
&\leq C\varepsilon M_\alpha(\nabla v)(X) + C\varepsilon^{-1}M_\alpha(\nabla_\eta v)(X),
\end{aligned} \tag{2.173}
$$

and thus

$$
M_\alpha(v)(X) \leq C\varepsilon M_\alpha(\nabla v)(X) + C\varepsilon^{-1}M_\alpha(\nabla_\eta v)(X), \tag{2.174}
$$

for every $X \in \partial\Omega$. Plugging this back in (2.171) and arranging that the term with small coefficient can be absorbed in the left-hand side, we finally arrive at (2.161). □

2.6 A singular integral operator related to the bi-Laplacian

Let Ω be as in (2.4). For $f \in L^p(\partial\Omega)$, $1 < p < \infty$, set

$$u(X) := \lim_{t \nearrow 1} \int_\Omega \Gamma(X - Y)\Big[2(n-2)\mathcal{D}f(tY) + 2(\nabla_\eta \mathcal{D}f)(tY)\Big]\, dY, \quad X \in \mathbb{R}^n, \tag{2.175}$$

where $\mathcal{D}$ is as in (2.118), and define the linear assignment

$$T : f \mapsto (\nabla_\eta u)\Big|_{\partial\Omega}. \tag{2.176}$$

To better understand the nature of this mapping, we shall derive an alternative representation, more amenable to the scope of the Calderón-Zygmund theory.

Before doing so, we make an observation of independent interest. Specifically,

$$u \text{ is as in (2.175)} \Longrightarrow u \text{ harmonic in } \Omega_- = \mathbb{R}^n \setminus \bar{\Omega}. \tag{2.177}$$

Indeed, this is going to be clear from (2.175) as soon as we show (see below) that the limit exists uniformly for X in compact subsets of Ω_-.

Turning now to the main task at hand, we first claim that for each fixed $X \in \Omega$ there exists $C(X) > 0$ with the property that

$$\sup_{t \in (1/2,1)} \left| \int_\Omega \Gamma(X - Y)\Big[2(n-2)\mathcal{D}f(tY) + 2(\nabla_\eta \mathcal{D}f)(tY)\Big]\, dY \right| \leq C(X)\|f\|_{L^p(\partial\Omega)}. \tag{2.178}$$

To justify this, decompose the domain of integration in $\int_\Omega \Gamma(X - Y)\mathcal{D}f(tY)\, dY$ into two pieces, $\Omega \setminus B(X, r)$ and $B(X, r)$, where $r := \operatorname{dist}(X, \partial\Omega)/2$. Note that for $Y \in \Omega \setminus B(X, r)$, the function $\Gamma(X - Y)$ is bounded by $C(X)$, whereas

$$\begin{aligned} \int_\Omega |\mathcal{D}f(tY)|\, dY &= t^{-n} \int_{t\Omega} |\mathcal{D}f(Z)|\, dZ \leq C\|\mathcal{D}f\|_{L^{np/(n-1)}(\Omega)} \\ &\leq C\|M(\mathcal{D}f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}. \end{aligned} \tag{2.179}$$

Also, straightforward estimates give $\sup_{t\in(1/2,1)} \sup_{Y \in B(X,r)} |\mathcal{D}f(tY)| \leq C(X)\|f\|_{L^p(\partial\Omega)}$, and the L^1-norm of $\Gamma(X - \cdot)$ on $B(X, r)$ is $\leq C(X)$. Altogether, the above reasoning shows that the contribution from $\int_\Omega \Gamma(X - Y)\mathcal{D}f)(tY)\, dY$ in (2.178) is of the right order.

Regarding the contribution from $\int_\Omega \Gamma(X - Y)(\nabla_\eta \mathcal{D}f)(tY)\, dY$, we integrate ∇_η by parts to re-write this as $\int_\Omega \partial_j^Y \Big[y_j \Gamma(X - Y)\Big]\mathcal{D}f(tY)\, dY$ plus $\int_{\partial\Omega} \Gamma(X - Y)y_j\nu_j(Y)\mathcal{D}f)(tY)\, d\sigma(Y)$. Then the first term is treated as above, while the second one is $\leq C(X)\|M(\mathcal{D}f)\|_{L^1(\partial\Omega)}$, uniformly in t. From this, (2.178) follows. We wish to point out that a reasoning based on similar ideas, and the identity (2.120), proves that

$$u \text{ is as in (2.175) with } f \in L^p_1(\partial\Omega) \tag{2.180}$$
$$\Longrightarrow u(X) = \int_\Omega \Gamma(X-Y)\Big[2(n-2)\mathcal{D}f(Y) + 2(\nabla_\eta \mathcal{D}f)(Y)\Big]\, dY, \quad X \in \mathbb{R}^n.$$

Having shown that the limit in (2.175) exists if $f \in L^p_1(\partial\Omega)$, $1 < p < \infty$, our next goal is to find a suitable representation for it. To this end, let B be as in (2.143), and recall (2.144). Thus, for each fixed $X \in \Omega$ and $t \in (0,1)$,

$$\int_\Omega \Gamma(X-Y)(\nabla_\eta \mathcal{D}f)(tY)\, dY = \int_\Omega \partial^Y_k \partial^Y_k \Big[B(X-Y)\Big] \nabla_\eta[\mathcal{D}f(tY)]\, dY. \tag{2.181}$$

Making use of (2.147), for each fixed $X \in \Omega$ we have

$$\begin{aligned}
&\lim_{t\nearrow 1} \int_\Omega \Gamma(X-Y)\nabla_\eta[\mathcal{D}f(tY)]\, dY \\
&\quad = \lim_{t\nearrow 1} \int_\Omega \nabla^Y_\eta \partial^Y_k \Big[B(X-Y)\Big] \partial_k[\mathcal{D}f(tY)]\, dY \\
&\quad\quad + (n-1) \lim_{t\nearrow 1} \int_\Omega \partial^Y_k \Big[B(X-Y)\Big] \partial_k[\mathcal{D}f(tY)]\, dY \\
&\quad\quad + \int_{\partial\Omega} \partial_{\tau_{jk}(Y)} \Big[y_j \partial^Y_k [B(X-Y)]\Big] (\tfrac{1}{2}I + K)f(Y)\, d\sigma(Y)
\end{aligned} \tag{2.182}$$

where, in the last boundary integral above, we have integrated by parts on $\partial\Omega$ and used the fact that $(\mathcal{D}f)(tY) \to (\frac{1}{2}I + K)f(Y)$ as $t \nearrow 1$ as functions in $L^p(\partial\Omega)$. Using the fact that $\partial^Y_{\tau_{jk}} y_j = (1-n)\nu_k(Y)$, this integral can be further transformed into

$$\begin{aligned}
&\int_{\partial\Omega} y_j \partial_{\tau_{jk}(Y)} \partial^Y_k \Big[B(X-Y)\Big] \Big(\tfrac{1}{2}I + K\Big) f(Y)\, d\sigma(Y) \\
&\quad - (n-1) \int_{\partial\Omega} \partial_{\nu(Y)} \Big[B(X-Y)\Big] \Big(\tfrac{1}{2}I + K\Big) f(Y)\, d\sigma(Y).
\end{aligned} \tag{2.183}$$

As for the first two terms in the right-hand side of (2.182), we use (2.29) in order to write

$$
\begin{aligned}
&\lim_{t\nearrow 1}\int_\Omega \nabla_\eta^Y \partial_k^Y \Big[B(X-Y)\Big]\partial_k[\mathcal{D}f(tY)]\,dY\\
&\qquad +(n-1)\lim_{t\nearrow 1}\int_\Omega \partial_k^Y\Big[B(X-Y)\Big]\partial_k[\mathcal{D}f(tY)]\,dY\\
&=\lim_{t\nearrow 1}\int_\Omega \partial_k^Y\nabla_\eta^Y\Big[B(X-Y)\Big]\partial_k[\mathcal{D}f(tY)]\,dY\\
&\qquad +(n-2)\lim_{t\nearrow 1}\int_\Omega \partial_k^Y\Big[B(X-Y)\Big]\partial_k[\mathcal{D}f(tY)]\,dY\\
&=: I+II. \qquad (2.184)
\end{aligned}
$$

Assume that $f\in L_1^p(\partial\Omega)$. This allows us to integrate ∂_k^Y by parts in II and use (2.122) to obtain

$$
\begin{aligned}
II &= (n-2)\lim_{t\nearrow 1}\int_{\partial\Omega} B(X-Y)\partial_{\nu(Y)}[\mathcal{D}f(tY)]\,d\sigma(Y)\\
&= \frac{n-2}{2}\int_{\partial\Omega} B(X-Y)\partial_{\tau_{jk}}S(\partial_{\tau_{jk}}f)(Y)\,d\sigma(Y) \qquad (2.185)\\
&= -\frac{n-2}{2}\int_{\partial\Omega}\partial_{\tau_{jk}(Y)}\Big[B(X-Y)\Big]S(\partial_{\tau_{jk}}f)(Y)\,d\sigma(Y).
\end{aligned}
$$

We now consider I in (2.184). Integrate ∂_k^Y by parts and, assuming that $f\in L_1^p(\partial\Omega)$, use (2.122) to transform it as

$$
\begin{aligned}
I &= \lim_{t\nearrow 1}\int_\Omega \partial_k^Y\nabla_\eta^Y\Big[B(X-Y)\Big]\partial_k[\mathcal{D}f(tY)]\,dY\\
&= -\frac{1}{2}\int_{\partial\Omega}\partial_{\tau_{jk}(Y)}\nabla_\eta^Y\Big[B(X-Y)\Big]S(\partial_{\tau_{jk}}f)(Y)\,d\sigma(Y). \qquad (2.186)
\end{aligned}
$$

Next, we note that

$$
\begin{aligned}
\lim_{t\nearrow 1}\int_\Omega \Gamma(X-Y)\mathcal{D}f(tY)\,dY &= \lim_{t\nearrow 1}\int_\Omega \Delta_Y\Big[B(X-Y)\Big]\mathcal{D}f(tY)\,dY \qquad (2.187)\\
&= \int_{\partial\Omega}\partial_{\nu(Y)}\Big[B(X-Y)\Big]\Big(\tfrac{1}{2}I+K\Big)f(Y)\,d\sigma(Y)\\
&\qquad -\int_{\partial\Omega} B(X-Y)\partial_\nu\mathcal{D}f(Y)\,d\sigma(Y) =: III+IV
\end{aligned}
$$

and, much as before,

$$IV = \frac{1}{2}\int_{\partial\Omega} \partial_{\tau_{jk}(Y)}\Big[B(X-Y)\Big] S(\partial_{\tau_{jk}} f)(Y)\, d\sigma(Y). \tag{2.188}$$

Consequently, for each $f \in L^p_1(\partial\Omega)$ and $X \in \Omega$, we have

$$\begin{aligned}\lim_{t\nearrow 1}\int_{\Omega} &\Gamma(X-Y)\mathcal{D}f(tY)\, dY \\ &= \int_{\partial\Omega} \partial_{\nu(Y)}\Big[B(X-Y)\Big]\Big(\tfrac{1}{2}I+K\Big)f(Y)\, d\sigma(Y) \\ &\quad +\frac{1}{2}\int_{\partial\Omega} \partial_{\tau_{jk}(Y)}\Big[B(X-Y)\Big] S(\partial_{\tau_{jk}} f)(Y)\, d\sigma(Y). \end{aligned}\tag{2.189}$$

After this preamble, it is straightforward to establish the following.

Lemma 2.13. *If Ω is as in (2.4) and $f \in L^p(\partial\Omega)$ with $1 < p < \infty$, then*

$$\begin{aligned} u(X) &:= \lim_{t\nearrow 1}\int_{\Omega}\Gamma(X-Y)\Big\{2(n-2)\mathcal{D}f(tY) + 2(\nabla_\eta \mathcal{D}f)(tY)\Big\}dY \\ &= 2\int_{\partial\Omega} y_j \partial_{\tau_{jk}(Y)}\partial^Y_k\Big[B(X-Y)\Big]\Big(\tfrac{1}{2}I+K\Big)f(Y)\, d\sigma(Y) \\ &\quad -\int_{\partial\Omega} \partial_{\tau_{jk}(Y)}\nabla^Y_\eta\Big[B(X-Y)\Big] S(\partial_{\tau_{jk}} f)(Y)\, d\sigma(Y) \\ &\quad -2\int_{\partial\Omega} \partial_{\nu(Y)}\Big[B(X-Y)\Big]\Big(\tfrac{1}{2}I+K\Big)f(Y)\, d\sigma(Y), \quad X \notin \partial\Omega. \end{aligned}\tag{2.190}$$

Proof. This is an immediate consequence of (2.178), the existence of the limit of u in (2.175) when $f \in L^p_1(\partial\Omega)$, plus its corresponding formula (implicit above), and a density argument. That the latter works is guaranteed by the fact that the boundary integrals in (2.190) can be estimated by $C(X)\|f\|_{L^p(\partial\Omega)}$. $\square$

As a corollary of this integral representation we have:

Proposition 2.14. *Let Ω be as in (2.4). The following properties hold:*

(1) $\|M(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}$, $1 < p < \infty$;

(2) $\|M(\nabla\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{h^{1,p}(\partial\Omega)}$, $\frac{n-1}{n} < p < \infty$;

(3) $u\Big|_{\partial\Omega_+} = u\Big|_{\partial\Omega_-}$ *and* $\nabla u\Big|_{\partial\Omega_+} = \nabla u\Big|_{\partial\Omega_-}$;

(4) The mapping $T : f \mapsto (\nabla_\eta u)\Big|_{\partial\Omega}$ *is well defined and bounded on* $L^p(\partial\Omega)$ *for* $1 < p < \infty$, *on* $L^p_1(\partial\Omega)$ *for* $1 < p < \infty$, *as well as on* $h^{1,p}_{at}(\partial\Omega)$ *for* $\frac{n-1}{n} < p \leq 1$;

(5) The following representation of T in terms of boundary integral operators is valid:

$$
\begin{aligned}
Tf(X) \;=\; & -2p.v.\int_{\partial\Omega} x_i y_j \partial_{\tau_{jk}(Y)}\partial_i^Y\partial_k^Y\Big[B(X-Y)\Big]\Big(\tfrac{1}{2}I+K\Big)f(Y)\,d\sigma(Y) \\
& +p.v.\int_{\partial\Omega} x_i \partial_{\tau_{jk}(Y)}\nabla_\eta^Y\partial_i^Y\Big[B(X-Y)\Big]S(\partial_{\tau_{jk}}f)(Y)\,d\sigma(Y) \qquad (2.191) \\
& +2\int_{\partial\Omega} x_i\partial_{\nu(Y)}\partial_i^Y\Big[B(X-Y)\Big]\Big(\tfrac{1}{2}I+K\Big)f(Y)\,d\sigma(Y), \quad X\in\partial\Omega.
\end{aligned}
$$

Proof. The proof of (1) follows from the fact that $\nabla^3 B$ is of Calderón-Zygmund type, while (2) is due to the presence of ∇_τ, i.e., the integral kernel of the top singular terms is of the form $\partial_\tau\nabla^2 B$. □

Next we take up the task of finding a suitable representation for T^*, the formal adjoint of T in (2.175)-(2.176).

Proposition 2.15. *Let Ω be as in (2.4). Then for every $f\in L^p(\partial\Omega)$, with $1<p<\infty$, there holds*

$$
\begin{aligned}
T^*f(X) \;=\; & -2\Big(\tfrac{1}{2}I+K^*\Big)^Z\Big(z_j\partial^Z_{\tau_{jk}}\int_{\partial\Omega}\partial_i^Z\partial_k^Z\Big[B(Z-Y)\Big]y_i f(Y)\,d\sigma(Y)\Big)(X) \\
& -\partial_{\tau_{jk}}S^Z\Big(z_r\partial^Z_{\tau_{jk}}\int_{\partial\Omega}\partial_r^Z\partial_i^Z\Big[B(Z-Y)\Big]y_i f(Y)\,d\sigma(Y)\Big)(X) \qquad (2.192) \\
& +2\Big(\tfrac{1}{2}I+K^*\Big)^Z\Big(\int_{\partial\Omega}\partial_{\nu(Z)}\partial_i^Z\Big[B(Z-Y)\Big]y_i f(Y)\,d\sigma(Y)\Big)(X),
\end{aligned}
$$

with the following convention in place: if $\mathcal{T}$ is a singular integral operator with kernel $k(X,Y)$, then

$$
\mathcal{T}^Z(g(Z))(X) := \int_{\partial\Omega} k(X,Z)g(Z)\,d\sigma(Z). \qquad (2.193)
$$

Proof. This follows from (2.191) and straightforward algebraic manipulations. □

Our next result is a direct consequence of Proposition 2.15 and Calderón-Zygmund theory for singular integral operators on Lipschitz surfaces.

Lemma 2.16. *Let Ω be as in (2.4). Then the operator T^* from (2.192) induces a mapping*

$$
T^*: h^p_{at}(\partial\Omega)\to h^p_{at}(\partial\Omega) \qquad (2.194)
$$

which is well-defined, linear and bounded for each $p\in(\frac{n-1}{n},1]$.

Recall (2.140). With this piece of notation we have

$$Tf := (\nabla_\eta u)\Big|_{\partial\Omega} \quad \text{where} \quad u := \Gamma\Big(2(n-2)\mathcal{D}f + 2\nabla_\eta \mathcal{D}f\Big) \ \text{ in } \ \Omega, \tag{2.195}$$

where u is understood in the sense of (2.175). We remind the reader that ∂_ν^- stands for the normal derivative computed by approaching the boundary from inside Ω_-.

Lemma 2.17. *Assume that Ω is as in (2.4). Then for every $g \in L^p(\partial\Omega)$, $1 < p < \infty$, there holds*

$$(T^*g)(X) = 2\partial^-_{\nu(X)}\langle X, \nabla\Gamma(div\,\mathcal{S}(g\eta))(X)\rangle, \qquad X \in \partial\Omega. \tag{2.196}$$

Proof. Fix $f \in L^{p'}(\partial\Omega)$, $1/p + 1/p' = 1$. Using (2.195) we may write

$$\begin{aligned}
\int_{\partial\Omega} fT^*g\,d\sigma &= \int_{\partial\Omega}(Tf)g\,d\sigma \qquad\qquad (2.197)\\
&= \int_{\partial\Omega}\Big(\lim_{t\nearrow 1}\int_\Omega \langle X, (\nabla\Gamma)(X-Y)\rangle\Big\{2(n-2)\mathcal{D}f(Y) + 2\langle Y, \nabla\mathcal{D}f(tY)\rangle\Big\}dY\Big)g(X)\,d\sigma(X)\\
&= \lim_{t\nearrow 1}\int_\Omega\Big\{2(n-2)\mathcal{D}f(Y) + 2\langle Y, \nabla\mathcal{D}f(tY)\rangle\Big\}\Big(\int_{\partial\Omega}\langle X, (\nabla\Gamma)(X-Y)\rangle g(X)\,d\sigma(X)\Big)dY\\
&= \lim_{t\nearrow 1}\int_\Omega\Big\{2(n-2)\mathcal{D}f(Y) + 2\langle Y, \nabla\mathcal{D}f(tY)\rangle\Big\}\,\mathrm{div}\,\mathcal{S}(g\eta)(Y)dY\\
&= \lim_{t\nearrow 1}\int_\Omega\Big(\int_{\partial\Omega}\Big\{2(n-2)\partial_{\nu(Y)}[\Gamma(X-Y)] + 2\langle X, \partial_{\nu(Y)}[(\nabla\Gamma)(tX-Y)]\rangle\Big\}f(Y)\,d\sigma(Y)\Big)\times\\
&\qquad\qquad \times\,\mathrm{div}\,\mathcal{S}(g\eta)(X)\,dX\\
&= \int_{\partial\Omega} f(Y)\Big(\lim_{t\nearrow 1}\int_\Omega\Big\{2(n-2)\partial_{\nu(Y)}[\Gamma(X-Y)] + 2\langle X, \partial_{\nu(Y)}[(\nabla\Gamma)(tX-Y)]\rangle\Big\}\times\\
&\qquad\qquad \times\,\mathrm{div}\,\mathcal{S}(g\eta)(X)\,dX\Big)d\sigma(Y).
\end{aligned}$$

Thus, since f was arbitrary,

$$\begin{aligned}
(T^*g)(X) \ &= \ 2(n-2)\partial_{\nu(X)}\Gamma(\mathrm{div}\,\mathcal{S}(g\eta))(X)\\
&\quad -2\lim_{t\nearrow 1}\partial_{\nu(X)}\int_\Omega\langle Y, (\nabla\Gamma)(X-tY)\rangle\,\mathrm{div}\,\mathcal{S}(g\eta)(Y)\,dY.
\end{aligned} \tag{2.198}$$

Now we write

$$\langle Y, (\nabla\Gamma)(X-tY)\rangle = t^{-1}(n-2)\Gamma(X-tY) + t^{-1}\langle X, (\nabla\Gamma)(X-tY)\rangle \tag{2.199}$$

and note that

$$(\nabla\Gamma)(X-tY)=(\nabla\Gamma)(X/t-Y)\cdot t^{1-n}\longrightarrow(\nabla\Gamma)(X-Y)\quad\text{as } t\nearrow 1, \tag{2.200}$$

and $X/t\to X$ as $t\nearrow 1$, nontangentially from the *exterior* of Ω. Using these observations back in (2.198) we may then conclude that (2.196) holds. $\Box$

Recall now the operators R, R^* and $\mathcal{R}$ from (2.112), (2.113) and (2) in Lemma 2.6. Given two functions $f\in L^2_1(\partial\Omega)$ and $g\in L^2(\partial\Omega)$, set

$$u(X):=\int_\Omega\Gamma(X-Y)\Big[2(n-2)\mathcal{D}f(Y)+2\nabla_\eta\mathcal{D}f(Y)\Big]dY, \tag{2.201}$$

and compute

$$\begin{aligned}\int_{\partial\Omega}g\,Tf\,d\sigma &= \int_{\partial\Omega}g(X)\langle X,\nabla u(X)\rangle\,d\sigma(X) \qquad (2.202)\\ &= \int_{\partial\Omega}g(X)\Big(\int_\Omega\langle X,(\nabla\Gamma)(X-Y)\rangle\Big[2(n-2)\mathcal{D}f(Y)+2\nabla_\eta\mathcal{D}f(Y)\Big]dY\Big)d\sigma(X)\\ &= \int_\Omega\Big(\int_{\partial\Omega}\langle X,(\nabla\Gamma)(X-Y)\rangle g(X)\,d\sigma(X)\Big)\Big[2(n-2)\mathcal{D}f(Y)+2\nabla_\eta\mathcal{D}f(Y)\Big]dY\\ &= \int_\Omega(\mathcal{R}g)(X)\Big[2(n-2)\mathcal{D}f(X)+2\nabla_\eta\mathcal{D}f(X)\Big]dX,\end{aligned}$$

i.e.,

$$\int_{\partial\Omega}g\,Tf\,d\sigma=\int_\Omega\mathcal{R}g\Big\{2(n-2)\mathcal{D}f+2\nabla_\eta\mathcal{D}f\Big\}\,dX. \tag{2.203}$$

To proceed, recall the Poisson integral operator (2.125) and, for each $f\in L^2(\partial\Omega)$ set

$$\widetilde{Q}f(X):=\int_0^1\{(n-2)\mathscr{P}f(tX)+\langle tX,\nabla\mathscr{P}f(tX)\rangle\}t^{\frac{n}{2}-1}dt,\quad X\in\Omega. \tag{2.204}$$

Also, define

$$Qf:=(\widetilde{Q}f)\Big|_{\partial\Omega}. \tag{2.205}$$

We now record a useful result, due to G. Verchota [57].

Lemma 2.18. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a Lipschitz, starlike domain with respect to the origin. Then*

$$Q : L^2(\partial\Omega) \to L^2(\partial\Omega) \quad \textit{isomorphically}, \tag{2.206}$$

$$\Delta \widetilde{Q} f = 0 \quad \textit{in} \;\; \Omega, \tag{2.207}$$

$$\|M(\widetilde{Q}f)\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}, \tag{2.208}$$

and

$$n\,\widetilde{Q}f + 2\nabla_\eta \widetilde{Q}f = 2(n-2)\mathscr{P}f + 2\nabla_\eta \mathscr{P}f. \tag{2.209}$$

After this preamble, we now return to (2.203). Then

$$\begin{aligned} 2(n-2)\mathcal{D}f + 2\nabla_\eta \mathcal{D}f &= 2(n-2)\mathscr{P} \circ (\tfrac{1}{2}I + K)f + 2\nabla_\eta\big[\mathscr{P} \circ (\tfrac{1}{2}I + K)\big]f \\ &= n\widetilde{Q}\big((\tfrac{1}{2}I + K)f\big) + 2\nabla_\eta \widetilde{Q}\big((\tfrac{1}{2}I + K)f\big), \end{aligned} \tag{2.210}$$

where for the last equality in (2.210) we have applied Lemma 2.18. Further specialize matters to the case when

$$g = -\big(\tfrac{1}{2}\langle \eta, \nu\rangle I + R^*\big)^{-1} Q\big(\tfrac{1}{2}I + K\big)f. \tag{2.211}$$

This ensures that $\mathcal{R}g\Big|_{\partial\Omega} = \Big[\widetilde{Q}\big((\tfrac{1}{2}I + K)f\big)\Big]\Big|_{\partial\Omega}$ and since $\mathcal{R}g$, $\widetilde{Q}((\tfrac{1}{2}I + K)f)$ are harmonic in Ω and satisfy $M(\mathcal{R}g)$, $M(\widetilde{Q}(\tfrac{1}{2}I + K)f) \in L^2(\partial\Omega)$, by the uniqueness in the L^2-Dirichlet boundary value problem for the Laplacian in the Lipschitz Ω we obtain

$$\mathcal{R}g = \widetilde{Q}\big((\tfrac{1}{2}I + K)f\big) \quad \text{in } \Omega. \tag{2.212}$$

Combining (2.203), (2.210), (2.211), and (2.212) we obtain

$$\begin{aligned} \int_{\partial\Omega} gTf\, d\sigma &= \int_\Omega \widetilde{Q}\big((\tfrac{1}{2}I + K)f\big)\Big\{n\widetilde{Q}\big((\tfrac{1}{2}I + K)f\big) + 2\nabla_\eta \widetilde{Q}\big((\tfrac{1}{2}I + K)f\big)\Big\}\, dX \\ &= \int_\Omega \operatorname{div}\Big\{\Big[\widetilde{Q}\big((\tfrac{1}{2}I + K)f\big)\Big]^2 \eta\Big\}\, dX. \end{aligned} \tag{2.213}$$

Hence, upon observing that, in general

$$\operatorname{div}(u^2\eta) = nu^2 + 2u\nabla_\eta u, \tag{2.214}$$

the Divergence Theorem gives

$$\int_{\partial\Omega} gTf\,d\sigma = \int_{\partial\Omega} \langle\eta,\nu\rangle \Big|Q(\tfrac{1}{2}I+K)f\Big|^2\,d\sigma, \tag{2.215}$$

if g is as in (2.211). The computation leading up to (2.215) is most easily justified by assuming that $f \in L^2_1(\partial\Omega)$. However, a standard density argument then shows that (2.215) is valid for every $f \in L^2(\partial\Omega))$ if, once again, g is as in (2.211).

Since $\langle\eta,\nu\rangle \geq \kappa > 0$ a.e. on $\partial\Omega$ and both Q and $\frac{1}{2}I+K$ are invertible on $L^2(\partial\Omega)$, we obtain from (2.215) that

$$\int_{\partial\Omega} Af\,Tf\,d\sigma \geq C\int_{\partial\Omega} |f|^2\,d\sigma, \quad \forall\, f \in L^2(\partial\Omega), \tag{2.216}$$

where

$$A := -\big(\tfrac{1}{2}\langle\eta,\nu\rangle I + R^*\big)^{-1} \circ Q \circ \big(\tfrac{1}{2}I+K\big) \tag{2.217}$$

is an invertible operator on $L^2(\partial\Omega)$. Estimate (2.216) implies that $A^* \circ T$ is an accretive operator on $L^2(\partial\Omega)$ and, hence, is invertible. Since A^* is invertible on $L^2(\partial\Omega)$, we arrive at the conclusion that

$$T : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \quad \text{isomorphically.} \tag{2.218}$$

Known perturbation techniques (cf. the discussion in [23]) then yield the following.

Theorem 2.19. *In the context of (2.4), there exists* $\varepsilon = \varepsilon(\Omega) > 0$ *with the property that*

$$\begin{aligned} T &: L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \\ T^* &: L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \end{aligned} \tag{2.219}$$

isomorphically whenever $2-\varepsilon < p < 2+\varepsilon$.

3 The Dirichlet and regularity problems

3.1 The Dirichlet problem for $|p-2| < \varepsilon$

For Ω as in (2.4) and $1 < p < \infty$, consider the following problem

$$
\begin{cases}
\Delta^2 u = 0 \text{ in } \Omega, \\
u\Big|_{\partial\Omega} = g_0 \text{ on } \partial\Omega, \\
\partial_\nu u\Big|_{\partial\Omega} = g_1 \text{ on } \partial\Omega,
\end{cases}
\tag{3.1}
$$

where a solution is sought in the class of functions satisfying

$$
M(\nabla u) \in L^p(\partial\Omega). \tag{3.2}
$$

We have:

Theorem 3.1. *For every domain Ω as in (2.4) there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that if $2 - \varepsilon < p < 2 + \varepsilon$ then the boundary value problem (3.1)-(3.2) is uniquely solvable for any $g_0 \in L^p_1(\partial\Omega)$ and $g_1 \in L^p(\partial\Omega)$. Furthermore, the solution can be represented in the form*

$$
u = \mathcal{D}g_0 - \mathcal{S}g_1 - \Gamma\Big(2(n-2)\mathcal{D}f + 2\nabla_\eta \mathcal{D}f\Big) \quad in \quad \Omega, \tag{3.3}
$$

where the solid integral is to be understood in the sense of (2.175) and

$$
f := T^{-1}\Big(\nabla_\eta \Big[\mathcal{D}g_0 - \mathcal{S}g_1\Big]\Big|_{\partial\Omega_-}\Big). \tag{3.4}
$$

Moreover,

$$
\|u\|_{B^{p,2}_{1+\frac{1}{p}}(\Omega)} \leq C\|g_0\|_{L^p_1(\partial\Omega)} + C\|g_1\|_{L^p(\partial\Omega)} \qquad if \;\; 2-\varepsilon < p \leq 2, \tag{3.5}
$$

$$
\|u\|_{F^{p,q}_{1+\frac{1}{p}}(\Omega)} \leq C\|g_0\|_{L^p_1(\partial\Omega)} + C\|g_1\|_{L^p(\partial\Omega)} \qquad if \;\; 2 \leq p < 2+\varepsilon, \;\; 0 < q < \infty. \tag{3.6}
$$

Proof. Let f be such that

$$
Tf = (\nabla_\eta h)\Big|_{\partial\Omega_-}, \tag{3.7}
$$

where

$$
h := \mathcal{D}g_0 - \mathcal{S}g_1. \tag{3.8}
$$

Also, set

$$w := \Gamma\Big(2(n-2)\mathcal{D}f + 2\nabla_\eta \mathcal{D}f\Big), \tag{3.9}$$

with the convention that this is understood as in (2.175).

Then $\nabla_\eta h$ and $\nabla_\eta w$ are harmonic in Ω_- with their maximal functions in $L^2(\partial\Omega)$. Having the same boundary trace implies, due to the uniqueness in the L^p-Dirichlet problem with p near 2, that

$$\nabla_\eta h = \nabla_\eta w \quad \text{in } \Omega_-. \tag{3.10}$$

From this and (2.33) we obtain that $h = w$ in Ω_-, modulo constants. Since they both decay at infinity we have, in fact, $h = w$ in Ω_-. Thus,

$$w\Big|_{\partial\Omega_+} = w\Big|_{\partial\Omega_-} = h\Big|_{\partial\Omega_-} + C = (-\tfrac{1}{2}I + K)g_0 - Sg_1, \tag{3.11}$$

and

$$\partial_\nu w\Big|_{\partial\Omega_+} = \partial_\nu w\Big|_{\partial\Omega_-} = \partial_\nu^- h = \partial_\nu \mathcal{D}g_0 - \partial_\nu^- \mathcal{S}g_1. \tag{3.12}$$

Therefore, the function $u := h - w$ solves the boundary value problem (3.1). Furthermore, by (1) of Proposition 2.14, (3.2) holds. Uniqueness is proved as in [43].

Finally, consider (3.5)-(3.6) for u as in (3.3), with $f \in L^p(\partial\Omega)$, satisfying $\|f\|_{L^p(\partial\Omega)} \le C\|g_0\|_{L^p_1(\partial\Omega)} + C\|g_1\|_{L^p(\partial\Omega)}$. The claim we make is that $u \in B^{p,p\vee 2}_{1+1/p}(\Omega)$, plus a natural estimate, where we set $p \vee 2 := \max\{p, 2\}$. Given that $u \in \operatorname{Ker}\Delta^2$, the estimates (3.5)-(3.6) will then follow from this claim and Theorem 4.16.

To prove that $u \in B^{p,p\vee 2}_{1+1/p}(\Omega)$, we first note that the piece corresponding to the Newtonian potential is of the right order, by (2.190) and Proposition 4.23. The latter result also gives that $\mathcal{S}g_1 \in B^{p,p\vee 2}_{1+1/p}(\Omega)$ and $\mathcal{D}g_0 \in B^{p,p\vee 2}_{1+1/p}(\Omega)$ (for the double layer, (2.120) and (4.93) are also used). □

We complement Theorem 3.1 with the following remarks.

Remark 1. Given an *arbitrary* bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \ge 3$, it is possible to show that the problem (3.1)-(3.2) has a unique solution whenever $2 - \varepsilon < p < 2 + \varepsilon$, where $\varepsilon = \varepsilon(\Omega) > 0$. This can be done using the technique of "wiggling" star-shaped Lipschitz domains near the boundary of Ω, and invoking Theorem 3.1. See § 3 in [56] and the discussion on pp. 126-127 in [13].

Remark 2. The *exterior* version of the problem (3.1)-(3.2) is also well-posed for p near 2. In this scenario, the decay condition (2.108) should be also imposed.

That this is indeed the case can be seen from Theorem 3.1, coupled with our results in § 2.4; cf. Propositions 2.3, 2.4, 2.5 in particular.

Remark 3. In the context of Theorem 3.1, one can use the well-posedness result in order to show that

$$\int_\Omega \delta(X)|\nabla\nabla u(X)|^2\,dX \leq C\|g_0\|_{L^2_1(\partial\Omega)} + C\|g_1\|_{L^2(\partial\Omega)}. \tag{3.13}$$

Indeed, (3.13) follows from (3.5) with $p = 2$ and Theorem 4.16 (cf., also (4.57)). As a corollary, there exists a finite constant $C = C(\Omega) > 0$ such that for any biharmonic function u in Ω there holds

$$\|\mathcal{A}(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|M(\nabla u)\|_{L^2(\partial\Omega)}, \tag{3.14}$$

where the area-function $\mathcal{A}$ is as in (2.163). Having established (3.14), a real variable argument (as in the last section of [12]) allows one to then conclude that for each $0 < p < \infty$ there exists $C = C(\Omega, p) > 0$ such that

$$\|\mathcal{A}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|M(\nabla u)\|_{L^p(\partial\Omega)}, \tag{3.15}$$

uniformly for u biharmonic in Ω. This has first been established in [42], via good-λ inequalities. See also [12] for a more general setting.

Remark 4. If, in addition of being a bounded Lipschitz domain in $\mathbb{R}^n$, Ω has the property that

$$\nu \in \mathrm{vmo}\,(\partial\Omega), \tag{3.16}$$

then the problem (3.1)-(3.2) is well-posed for every $p \in (1, \infty)$. In particular, this is the case if $\partial\Omega$ is a compact surface locally given by the graph of a Lipschitz function whose gradient lies in Sarason's space VMO.

The above statement extends a similar result of G. Verchota, proved in [56], for C^1 domains. The general outline of the proof of our claim is the same as in [56] in which the compactness results from [14] are substituted by those proved in [20].

3.2 Possible formulations of the regularity problem

In this subsection we study a variant of (3.1), designed so that data exhibiting higher order smoothness can be prescribed. This is commonly referred to as the regularity problem.

One of the most studied boundary value problems for the bi-Laplacian in a domain Ω reads

$$(B)\;\begin{cases} \Delta^2 u = 0 \;\text{ in }\; \Omega, \\ u\Big|_{\partial\Omega} = F \;\text{ on }\; \partial\Omega, \\ \partial_\nu u\Big|_{\partial\Omega} = G \;\text{ on }\; \partial\Omega. \end{cases} \tag{3.17}$$

For various purposes, it is convenient to rephrase (3.17) in a way which avoids the appearance of the unit normal in the formulation of the boundary conditions. Such a version reads

$$(B')\quad \begin{cases} \Delta^2 u = 0 \ \text{ in } \ \Omega, \\ u\Big|_{\partial\Omega} = f_0 \ \text{ on } \ \partial\Omega, \\ (\partial_j u)\Big|_{\partial\Omega} = f_j \ \text{ on } \ \partial\Omega, \ 1 \leq j \leq n, \end{cases} \tag{3.18}$$

where, necessarily, the family

$$\dot{f} := (f_0, f_1, \dots, f_n) \text{ satisfies the compatibility conditions in (2.21).} \tag{3.19}$$

Proposition 3.2. *The boundary value problem* (B) *is uniquely solvable if and only if* (B') *is uniquely solvable.*

Proof. Suppose that (B) is solvable and that $\dot{f} = (f_0, f_1, \dots, f_n)$ is a given Whitney array (i.e, (2.21) holds). If we set

$$F := f_0 \quad \text{and} \quad G := \nu_j f_j, \tag{3.20}$$

then the solution u of (B) for this boundary data actually solves (B') with data $(f_0, f_1, \dots, f_n)$. Indeed, for every j,

$$\begin{aligned} \partial_j u &= \nu_k(\nu_k\partial_j - \nu_j\partial_k)u + \nu_j\partial_\nu u = \nu_k(\partial_{\tau_{kj}} F) + \nu_j G \\ &= \nu_k\partial_{\tau_{kj}} f_0 + \nu_j\nu_k f_k = \nu_k(\nu_k f_j - \nu_j f_k) + \nu_j\nu_k f_k = f_j. \end{aligned} \tag{3.21}$$

Conversely, suppose that a solution for (B') exists and let F and G be given. We then set

$$f_0 := F \quad \text{and} \quad f_j := \nu_j G + \nu_k\partial_{\tau_{kj}} F \ \text{ for } \ 1 \leq j \leq n. \tag{3.22}$$

Then $(f_0, f_1, \dots, f_n)$ is a Whitney array since for every $1 \leq j, k \leq n$ we have

$$\begin{aligned} \nu_j f_k - \nu_k f_j &= \nu_j\nu_a\partial_{\tau_{ak}} F - \nu_k\nu_b\partial_{\tau_{bj}} F \\ &= [\nu_j\nu_a(\nu_a\partial_k - \nu_k\partial_a) - \nu_k\nu_b(\nu_b\partial_j - \nu_j\partial_b)]F = \partial_{\tau_{jk}} F = \partial_{\tau_{jk}} f_0. \end{aligned} \tag{3.23}$$

Moreover, if u solves (B') with data $(f_0, f_1, \dots, f_n)$ constructed from F, G as in (3.22) then $u\Big|_{\partial\Omega} = f_0 = F$ and

$$\partial_\nu u = \nu_j(\partial_j u)\Big|_{\partial\Omega} = \nu_j f_j = \nu_j(\nu_j G + \nu_k \partial_{\tau_{kj}} F) = G. \tag{3.24}$$

Finally, the fact that each of the problems (B), (B') enjoys uniqueness if the other does, is implicit in what we have proved so far. $\square$

Remark. It is implicit in the above considerations that the mappings

$$P(\dot{f}) := (f_0, \nu_j f_j) \quad \text{if} \quad \dot{f} = (f_0, f_1, ..., f_n), \quad \text{is a Whitney array,} \tag{3.25}$$

$$\text{and} \quad Q(F,G) := \dot{f}, \quad \text{where} \quad \dot{f} := \Big(F\,,\, (\nu_j G + \nu_k \partial_{\tau_{kj}} F)_{1\le j\le n}\Big) \tag{3.26}$$

are inverse to one another. Although not of crucial importance here, we whish to mention that

$$\begin{aligned} P &: WA\big(L^p(\partial\Omega)\big) \longrightarrow L^p_1(\partial\Omega) \oplus L^p(\partial\Omega), \\ Q &: L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \longrightarrow WA\big(L^p(\partial\Omega)\big), \end{aligned} \tag{3.27}$$

are isomorphism, for each $p \in (1,\infty)$.

Proceeding further, consider the following "nonstandard" version of the regularity problem:

$$(B'') \left\{ \begin{array}{l} \Delta^2 u = 0 \ \text{ in } \ \Omega, \\ \dfrac{\eta_j}{|\eta|}\partial_j u = g_0 \ \text{ on } \ \partial\Omega, \\ |\eta|^{n-3}\partial_{\tau_{jk}}\Big(\dfrac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big) = g_1 \ \text{ on } \ \partial\Omega. \end{array} \right. \tag{3.28}$$

For further reference, let us note here that

$$|\eta|^{n-3}\partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big) = |\eta|^{-1}\Big[\partial_\nu \nabla_\eta - \langle\nu,\eta\rangle\Delta + \partial_\nu\Big]u + (n-2)|\eta|^{-3}\langle\nu,\eta\rangle\nabla_\eta u. \tag{3.29}$$

3.3 First look at the nonstandard version

In this subsection we explore the extent to which (3.28) is equivalent to (3.18). This is accomplished in a series of propositions, starting with:

Proposition 3.3. *Suppose that Ω is as in (2.4), u is a real-valued function defined in $\bar{\Omega}$, and $\dot{f} = (f_0, f_1, ..., f_n)$ is a Whitney array on $\partial\Omega$ such that*

$$\eta_j(\partial_j u - f_j) = 0 \quad \text{on } \ \partial\Omega. \tag{3.30}$$

Then, for $X \in \partial\Omega$,

$$\partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)\Big)(X) \tag{3.31}$$
$$= \frac{-1}{\varphi(X/|X|)^{n-2}\sqrt{|(\nabla_{tan}\varphi)(X/|X|)|^2 + |\varphi(X/|X|)|^2}}[\Delta_{S^{n-1}}(\tilde{u} - \widetilde{f}_0)](X/|X|),$$

where $\tilde{u}$, $\widetilde{f}_0$ are the extensions of $u|_{\partial\Omega}$ and f_0, respectively, to $\mathbb{R}^n \setminus \{0\}$, as defined in (2.13).

Proof. Fix $\psi \in C_0^\infty(\mathbb{R}^n)$, and set

$$\widetilde{\psi}(X) := \psi\Big(\varphi(X/|X|)X/|X|\Big), \qquad \text{for each} \quad X \in \mathbb{R}^n \setminus \{0\}. \tag{3.32}$$

An integration by parts gives

$$\begin{aligned}\int_{\partial\Omega} \partial_{\tau_{jk}}\Big[\frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)\Big]\psi\, d\sigma &= -\int_{\partial\Omega} \frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)(\partial_{\tau_{jk}}\psi)\, d\sigma \\ &= -\int_{\partial\Omega} \frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)(\nu_j\partial_k\psi - \nu_k\partial_j\psi)\, d\sigma. \end{aligned} \tag{3.33}$$

Use now (2.8)-(2.9) to change variables from $\partial\Omega$ to S^{n-1}:

$$\begin{aligned}\int_{\partial\Omega} \partial_{\tau_{jk}}\Big[\frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)\Big]\psi\, d\sigma &= -\int_{S^{n-1}} \omega_k\varphi(\omega)\Big[(\partial_j u)(\varphi(\omega)\omega) - f_j(\varphi(\omega)\omega)\Big] \times \\ &\quad \times\Big\{\Big[\varphi(\omega)\omega_j - (\nabla_{tan}\varphi)_j(\omega)\Big](\partial_k\psi)(\varphi(\omega)\omega) \\ &\quad -\Big[\varphi(\omega)\omega_k - (\nabla_{tan}\varphi)_k(\omega)\Big](\partial_j\psi)(\varphi(\omega)\omega)\Big\}\, d\omega.\end{aligned} \tag{3.34}$$

Now, observe that $\omega_k(\nabla_{tan}\varphi)_k(\omega) = 0$ on S^{n-1}. Also, (3.30) gives

$$\omega_j\Big((\partial_j u)(\varphi(\omega)\omega) - f_j(\varphi(\omega)\omega)\Big) = 0, \qquad \omega \in S^{n-1}. \tag{3.35}$$

Using these, the right-hand side of (3.34) reduces to

$$\begin{aligned}&\int_{S^{n-1}} \varphi(\omega)\Big[(\partial_j u)(\varphi(\omega)\omega) - f_j(\varphi(\omega)\omega)\Big] \times \\ &\quad \times\Big\{(\nabla_{tan}\varphi)_j(\omega)\omega_k(\partial_k\psi)(\varphi(\omega)\omega) + \varphi(\omega)(\partial_j\psi)(\varphi(\omega)\omega)\Big\}\, d\omega.\end{aligned} \tag{3.36}$$

Next, (3.32) and the first equality in (2.15) give

$$(\partial_j\psi)(\varphi(\omega)\omega) = (\partial_j\widetilde{\psi})(\varphi(\omega)\omega) + \varphi(\omega)^{-1}(\varphi(\omega)\omega_j - (\nabla_{tan}\varphi)_j(\omega))\omega_k(\partial_k\psi)(\varphi(\omega)\omega). \quad (3.37)$$

Keeping this, as well as (3.35), in mind, (3.36) further simplifies to

$$\int\limits_{S^{n-1}} \varphi(\omega)^2\Big[(\partial_j u)(\varphi(\omega)\omega) - f_j(\varphi(\omega)\omega)\Big](\partial_j\widetilde{\psi})(\varphi(\omega)\omega)\,d\omega. \quad (3.38)$$

Formulas (2.15) and (2.23) also ensure that

$$(\partial_j u)(\varphi(\omega)\omega) = (\partial_j\tilde{u})(\varphi(\omega)\omega) + \frac{\nu_j(\varphi(\omega)\omega)}{\langle\omega,\nu(\varphi(\omega)\omega)\rangle}\omega_k(\partial_k u)(\varphi(\omega)\omega), \quad (3.39)$$

$$f_j(\varphi(\omega)\omega) = (\partial_j\widetilde{f}_0)(\varphi(\omega)\omega) + \frac{\nu_j(\varphi(\omega)\omega)}{\langle\omega,\nu(\varphi(\omega)\omega)\rangle}\omega_k f_k(\varphi(\omega)\omega). \quad (3.40)$$

Subtracting (3.40) from (3.39) and recalling (3.30), we obtain

$$(\partial_j u)(\varphi(\omega)\omega) - f_j(\varphi(\omega)\omega) = \partial_j(\tilde{u} - \widetilde{f}_0)(\varphi(\omega)\omega), \qquad \omega\in S^{n-1}. \quad (3.41)$$

Note that, by homogeneity, $\partial_j(\tilde{u} - \widetilde{f}_0)(\varphi(\omega)\omega), = \varphi(\omega)^{-1}\partial_j(\tilde{u} - \widetilde{f}_0)(\omega)$ and $(\partial_j\widetilde{\psi})(\varphi(\omega)\omega) = \varphi(\omega)^{-1}(\partial_j\widetilde{\psi})(\omega)$. This and a reference to (2.25) then allow us to re-write (3.38) as

$$\int\limits_{S^{n-1}} \partial_j(\tilde{u} - \widetilde{f}_0)(\omega)(\partial_j\widetilde{\psi})(\omega)\,d\omega = -\int\limits_{S^{n-1}} \Delta_{S^{n-1}}(\tilde{u} - \widetilde{f}_0)(\omega)\widetilde{\psi}(\omega)\,d\omega. \quad (3.42)$$

Recalling that this expression stands for the right-hand side of (3.34), yields

$$\begin{aligned}\int\limits_{\partial\Omega} \partial_{\tau_{jk}}\Big[\frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)\Big]\psi\,d\sigma &= -\int\limits_{S^{n-1}} \Delta_{S^{n-1}}(\tilde{u} - \widetilde{f}_0)(\omega)\,\widetilde{\psi}(\omega)\,d\omega \quad (3.43)\\ &= -\int\limits_{\partial\Omega} \frac{[\Delta_{S^{n-1}}(\tilde{u} - \widetilde{f}_0)](X/|X|)}{\varphi(X/|X|)^{n-2}\sqrt{|(\nabla_{tan}\varphi)(X/|X|)|^2 + |\varphi(X/|X|)|^2}}\psi(X)\,d\sigma(X),\end{aligned}$$

by (2.9). Since ψ is arbitrary, (3.31) follows. □

Our next result explains how one can treat (3.18) by solving (3.28) for data appropriately tailored out of the original data for (3.18).

Proposition 3.4. *Assume that u solves (3.28) when the data has the following special form:*

$$g_0 := |\eta|^{-1}\eta_j f_j, \qquad g_1 := |\eta|^{n-3}\partial_{\tau_{jk}}(\eta_k|\eta|^{2-n} f_j), \tag{3.44}$$

with $f_1, ..., f_n$ the last n components of a Whitney array $\dot{f} = (f_0, f_1, ..., f_n)$. Then

$$u = f_0 + C \quad \text{on } \partial\Omega, \text{ for some constant } C, \tag{3.45}$$

and

$$\partial_j u = f_j \quad \text{on } \partial\Omega, \text{ for each } j \in \{1, ..., n\}. \tag{3.46}$$

In particular, $u - C$ solves (3.18).

Proof. The special format of the data in (3.44) ensures that

$$\eta_j(\partial_j u - f_j) = 0 \quad \text{and} \quad \partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)\Big) = 0 \ \text{ on } \partial\Omega. \tag{3.47}$$

Thus, (3.45) follows by invoking Proposition 3.3 and (2.27). With this in hand, for an arbitrary $j \in \{1, ..., n\}$, we now compute

$$(\partial_j u)\Big|_{\partial\Omega} = \Big(\partial_j u - \frac{\nu_j}{\langle \nu, \eta\rangle}\langle \eta, \nabla u\rangle\Big) + \frac{\nu_j}{\langle \nu, \eta\rangle}\nabla_\eta u =: E_1 + E_2. \tag{3.48}$$

Now,

$$\begin{aligned} \langle \eta, \nu\rangle\, E_1 &= (\eta_k\nu_k\partial_j - \nu_j\eta_k\partial_k)u = \eta_k(\partial_{\tau_{kj}} u) \\ &= \eta_k(\partial_{\tau_{kj}} f_0) = \eta_k(\nu_k f_j - \nu_j f_k) = \langle \eta, \nu\rangle f_j - \nu_j\eta_k f_k, \end{aligned} \tag{3.49}$$

by (3.45) and (2.21). In addition, $\nabla_\eta u = \eta_k f_k$ by the first formula in (3.47), so that

$$\langle \eta, \nu\rangle\, E_2 = \nu_j\eta_k f_k. \tag{3.50}$$

Combining (3.48) with (3.49)-(3.50) now gives $(\partial_j u)|_{\partial\Omega} = f_j$, for every $j \in \{1, ..., n\}$. □

Compared to Proposition 3.4, our last result in this subsection goes in the opposite direction. Somewhat more specifically, here we explain how the solution of (3.28) with $g_0 = 0$ can be regarded as the solution of (3.18) for a certain type of data.

Proposition 3.5. *Assume that u solves (3.28) with*

$$g_0 \equiv 0 \quad \textit{and } g_1 \textit{ such that} \quad \int_{\partial\Omega} |X|^{3-n} g_1(X)\, d\sigma(X) = 0. \tag{3.51}$$

For $\omega \in S^{n-1}$ define

$$g(\omega) := -g_1(\varphi(\omega)\omega)\, \varphi(\omega) \sqrt{|(\nabla_{tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2} \tag{3.52}$$

then solve

$$\Delta_{S^{n-1}} w(\omega) = g(\omega), \qquad \omega \in S^{n-1}, \tag{3.53}$$

and finally set

$$u_o(X) := w(X/|X|), \qquad X \in \mathbb{R}^n \setminus \{0\}. \tag{3.54}$$

Then there exists a constant C such that

$$u\Big|_{\partial\Omega} = u_o\Big|_{\partial\Omega} + C \quad \textit{and} \quad (\partial_j u)\Big|_{\partial\Omega} = (\partial_j u_o)\Big|_{\partial\Omega} \quad \textit{for} \quad 1 \leq j \leq n. \tag{3.55}$$

Let us remark that the last condition in (3.51) implies, thanks to (2.9), that

$$\int_{S^{n-1}} g(\omega)\, d\omega = 0 \tag{3.56}$$

which, in turn, is necessary for the solvability of $\Delta_{S^{n-1}} w = g$ on S^{n-1}. Also, any two solutions of this latter equation differ only by a constant (cf. (2.27)).

Proof of Proposition 3.5. Using Proposition 3.3 in the particular case when the Whitney array in question is $(0, ..., 0)$, we obtain

$$\partial_{\tau_{jk}} \Big(\frac{\eta_k}{|\eta|^{n-2}} \partial_j u \Big)(X) \tag{3.57}$$

$$= -\frac{(\Delta_{S^{n-1}} \tilde{u})(X/|X|)}{\varphi(X/|X|)^{n-2} \sqrt{|(\nabla_{tan}\varphi)(X/|X|)|^2 + |\varphi(X/|X|)|^2}},$$

for $X \in \partial\Omega$.

Next, consider $\dot{f} = (f_0, f_1, ..., f_n)$ where

$$f_0 := u_o\Big|_{\partial\Omega} \text{ and } \quad f_j := (\partial_j u_o)\Big|_{\partial\Omega} \quad \text{for} \quad 1 \leq j \leq n. \tag{3.58}$$

Then $\dot{f}$ is a Whitney array on $\partial\Omega$ and

$$\eta_j f_j = (\nabla_\eta u_o)\Big|_{\partial\Omega} = 0, \tag{3.59}$$

since u_o is radially independent. Moreover, since $\widetilde{f}_0(\omega) = f_0(\varphi(\omega)\omega) = u_o(\varphi(\omega)\omega) = w(\omega)$ we obtain $\Delta_{S^{n-1}}\widetilde{f}_0(\omega) = \Delta_{S^{n-1}}w(\omega) = g(\omega)$ if $\omega \in S^{n-1}$. This, (3.52) and the fact that u is a solution then give

$$\begin{aligned} -\frac{(\Delta_{S^{n-1}}\widetilde{f}_0)(X/|X|)}{\varphi(X/|X|)^{n-2}\sqrt{|(\nabla_{tan}\varphi)(X/|X|)|^2 + |\varphi(X/|X|)|^2}} &= |X|^{3-n}g_1(X) \\ &= \partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big)(X), \end{aligned} \tag{3.60}$$

for $X \in \partial\Omega$ (since $X = \varphi(X/|X|)X/|X|$ and $\varphi(X/|X|) = |X|$ in this case). By subtracting (3.60) from (3.57) and invoking (3.31) we then arrive at the conclusion that

$$\partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}(\partial_j u - f_j)\Big) = 0 \qquad \text{on} \quad \partial\Omega. \tag{3.61}$$

Hence, (3.47) holds for $\dot{f}$ as in (3.58). Having established this, (3.55) follows from (the proof of) Proposition 3.4. $\square$

3.4 The regularity problem with p near 2

For $1 < p < \infty$, set

$$L^p_\eta(\partial\Omega) := \Big\{f \in L^p(\partial\Omega) : \int_{\partial\Omega} |X|^{3-n} f(X)\, d\sigma(X) = 0\Big\}. \tag{3.62}$$

For latter purposes we also define

$$L^p_0(\partial\Omega) := \Big\{f \in L^p(\partial\Omega) : \int_{\partial\Omega} f(X)\, d\sigma(X) = 0\Big\}. \tag{3.63}$$

Obviously, $L^p_\eta(\partial\Omega) = L^p_0(\partial\Omega)$ when $n = 3$. The main result of this subsection reads as follows:

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that, for each $p \in (2-\varepsilon, 2+\varepsilon)$, the problem*

$$
(B'')\quad \begin{cases} \Delta^2 u = 0 \quad in \ \Omega, \\ \dfrac{\eta_j}{|\eta|}(\partial_j u)\Big|_{\partial\Omega} = g_0 \in L^p_1(\partial\Omega), \\ |\eta|^{n-3}\partial_{\tau_{jk}}\Big(\dfrac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big)\Big|_{\partial\Omega} = g_1 \in L^p_\eta(\partial\Omega), \\ M(\nabla\nabla u) \in L^p(\partial\Omega), \quad u(0) = 0, \end{cases} \tag{3.64}
$$

has a unique solution. This solution can be represented as

$$
u(X) = H(X) - 2\Gamma(\nabla_\eta \mathcal{S} f)(X) + C_1 + C_2|X|^2, \qquad X \in \Omega, \tag{3.65}
$$

where

$$
H(X) := \int_0^1 \widetilde{h}(tX)\frac{dt}{t}, \quad X \in \Omega, \tag{3.66}
$$

$$
\widetilde{h} := h - h(0), \qquad h := \mathcal{D} f_1 - \mathcal{S} f_2, \tag{3.67}
$$

$$
f_1 \in L^p_1(\partial\Omega), \quad f_2 \in L^p(\partial\Omega), \quad f \in L^p(\partial\Omega), \tag{3.68}
$$

$$
C_1 := 2\Gamma(\nabla_\eta \mathcal{S} f)(0), \quad C_2 \in \mathbb{R}. \tag{3.69}
$$

Furthermore,

$$
\|u\|_{B^{p,2}_{2+\frac{1}{p}}(\Omega)} \leq C\|g_0\|_{L^p_1(\partial\Omega)} + C\|g_1\|_{L^p(\partial\Omega)} \quad if \ \ 2-\varepsilon < p \leq 2, \tag{3.70}
$$

$$
\|u\|_{F^{p,q}_{2+\frac{1}{p}}(\Omega)} \leq C\|g_0\|_{L^p_1(\partial\Omega)} + C\|g_1\|_{L^p(\partial\Omega)} \quad if \ \ 2 \leq p < 2+\varepsilon, \ \ 0 < q < \infty. \tag{3.71}
$$

For each $p > \frac{n-1}{2}$, the membership $M(\nabla\nabla u) \in L^p(\partial\Omega)$ further implies

$$
u \in F^{\frac{np}{n-1},2}_2(\Omega) \hookrightarrow C^{2-\frac{n-1}{p}}(\bar{\Omega}). \tag{3.72}
$$

In particular, the solution u of (3.64) satisfies

$$
\begin{cases} u \in C^{1+\theta}(\bar{\Omega}) & \text{if } n = 3, \\ u \in C^{\frac{1}{2}+\theta}(\bar{\Omega}) & \text{if } n = 4, \\ u \in C^{\theta}(\bar{\Omega}) & \text{if } n = 5, \end{cases} \qquad \text{for some } \theta > 0. \tag{3.73}
$$

Proof of Theorem 3.6. Assume that u is as in (3.65), with the conventions in (3.66)-(3.69), *in which we take* $C_2 = 0$. Then $u(0) = \Delta u(0) = 0$ and, by Lemma 2.10, $M(\nabla\nabla u) \in L^p(\partial\Omega)$. Next, we compute

$$\langle X, (\nabla u)(X)\rangle = \widetilde{h}(X) - 2\langle X, \nabla\Gamma(\nabla_\eta \mathcal{S} f)(X)\rangle, \qquad X \in \Omega. \tag{3.74}$$

While u in (3.65) is only defined in Ω, the right-hand side of (3.74) turns out to be meaningful in Ω_- as well. Keeping this in mind, we may compute on $\partial\Omega_-$

$$\begin{aligned}
&\partial_\nu^- \Big[\widetilde{h}(X) - 2\langle X, \nabla\Gamma(\nabla_\eta \mathcal{S} f)(X)\rangle\Big] \\
&\quad = \partial_\nu^- h + \partial_\nu^- \Big[-2\langle X, \nabla\Gamma(\operatorname{div} \mathcal{S}(f\eta))(X)\rangle - 2(2-n)\langle X, \nabla\Gamma(\mathcal{S} f)(X)\rangle\Big] \\
&\quad = \partial_\nu^- h - T^* f + \mathscr{C}_1 f,
\end{aligned} \tag{3.75}$$

where we have set

$$\mathscr{C}_1 f(X) := 2(n-2)\partial_{\nu(X)}^- \Big[\langle X, \Gamma(\nabla \mathcal{S} f)(X)\rangle - \langle X, \mathcal{S}(\nu\, S f)(X)\rangle\Big], \qquad X \in \partial\Omega. \tag{3.76}$$

Above, we have made use of (2.34)-(2.36), the identity in (8) of Lemma 2.6, (2.196) and an integration by parts. Next, since by (3.67), (2.141) and (2.37),

$$\Delta\Big[\widetilde{h}(X) - 2\langle X, \nabla\Gamma(\nabla_\eta \mathcal{S} f)(X)\rangle\Big] = 0 \quad \text{in } \ \Omega_-, \tag{3.77}$$

uniqueness in the exterior Neumann problem for harmonic functions, plus the associated integral representation formula, gives

$$\begin{aligned}
&\widetilde{h}(X) - 2\langle X, \nabla\Gamma(\nabla_\eta \mathcal{S} f)(X)\rangle \\
&\qquad = \mathcal{S}\Big((\tfrac{1}{2}I + K^*)^{-1}(\partial_\nu^- h - T^* f + \mathscr{C}_1 f)\Big)(X) - h(0) \qquad \text{in } \ \Omega_-.
\end{aligned} \tag{3.78}$$

Consequently,

$$\begin{aligned}
(\nabla_\eta u)\Big|_{\partial\Omega_+} &= \Big[\widetilde{h} - 2\nabla_\eta\Gamma(\nabla_\eta \mathcal{S} f)\Big]\Big|_{\partial\Omega_+} \\
&= \Big[\widetilde{h} - 2\nabla_\eta\Gamma(\nabla_\eta \mathcal{S} f)\Big]\Big|_{\partial\Omega_-} + \text{jump}\Big(\widetilde{h} - 2\nabla_\eta\Gamma(\nabla_\eta \mathcal{S} f)\Big) \\
&= S\Big((\tfrac{1}{2}I + K^*)^{-1}(\partial_\nu^- h - T^* f + \mathscr{C}_1 f)\Big) - h(0) + f_1,
\end{aligned} \tag{3.79}$$

since the jump of $\widetilde{h}$ is f_1, and $\nabla_\eta\Gamma(\nabla_\eta \mathcal{S} f)$ does not jump across $\partial\Omega$.

Consider now the second boundary condition in (3.64) when u is as in (3.65)-(3.69) *where once again we take* $C_2 = 0$. We have

$$
\begin{aligned}
&|\eta|^{n-3}\partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big)\\
&\quad = |\eta|^{-1}\Big[\partial_\nu\nabla_\eta - \langle\nu,\eta\rangle\Delta + \partial_\nu\Big]H + (n-2)|\eta|^{-3}\langle\nu,\eta\rangle\nabla_\eta H\\
&\qquad -2|\eta|^{n-3}\partial^+_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j\Gamma(\nabla_\eta\mathcal{S}f)\Big)\\
&\quad = |\eta|^{-1}\Big[\partial^+_\nu h + \partial_\nu H\Big] + (n-2)|\eta|^{-3}\langle\nu,\eta\rangle\widetilde{h}\\
&\qquad -2|\eta|^{n-3}\partial^-_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j\Gamma(\nabla_\eta\mathcal{S}f)\Big),
\end{aligned}
\tag{3.80}
$$

since $\partial_j\Gamma(\nabla_\eta\mathcal{S}f)$ does not jump across $\partial\Omega$. Furthermore,

$$
\begin{aligned}
&-2|\eta|^{n-3}\partial^-_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j\Gamma(\nabla_\eta\mathcal{S}f)\Big)\\
&\quad = -2|\eta|^{-1}\Big[\partial^-_\nu\nabla_\eta - \langle\nu,\eta\rangle\Delta^- + \partial_\nu\Big]\Gamma(\nabla_\eta\mathcal{S}f)\\
&\qquad -2(n-2)|\eta|^{-3}\langle\nu,\eta\rangle\nabla_\eta\Gamma(\nabla_\eta\mathcal{S}f)\\
&\quad = -2|\eta|^{-1}\Big[\partial^-_\nu\nabla_\eta\Gamma(\nabla_\eta\mathcal{S}f) + \partial_\nu\Gamma(\nabla_\eta\mathcal{S}f)\Big]\\
&\qquad -2(n-2)|\eta|^{-3}\langle\nu,\eta\rangle\nabla_\eta\Gamma(\nabla_\eta\mathcal{S}f)\\
&\quad = -|\eta|^{-1}T^*f + \mathscr{C}_2 f,
\end{aligned}
\tag{3.81}
$$

where

$$
\begin{aligned}
\mathscr{C}_2 f \;:=\;& -2|\eta|^{-1}\partial_\nu\Gamma(\nabla_\eta\mathcal{S}f) - 2(n-2)|\eta|^{-3}\langle\nu,\eta\rangle\nabla_\eta\Gamma(\nabla_\eta\mathcal{S}f)\\
& -2(2-n)|\eta|^{-1}\partial^-_\nu\nabla_\eta\Gamma(\mathcal{S}f).
\end{aligned}
\tag{3.82}
$$

Combining (3.80-(3.82) we therefore obtain

$$
\begin{aligned}
&|\eta|^{n-3}\partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big)\\
&\quad = |\eta|^{-1}\Big[\partial^+_\nu h - T^*f\Big] + |\eta|^{-1}\partial_\nu H + (n-2)|\eta|^{-3}\langle\nu,\eta\rangle\widetilde{h} + \mathscr{C}_2 f \qquad (3.83)\\
&\quad = |\eta|^{-1}\Big[\partial^-_\nu h - T^*f\Big] + |\eta|^{-1}f_2 + |\eta|^{-1}\partial_\nu H + (n-2)|\eta|^{-3}\langle\nu,\eta\rangle\widetilde{h} + \mathscr{C}_2 f,
\end{aligned}
$$

since $\partial^+_\nu h - \partial^-_\nu h = f_2$.

Having established the trace formulas (3.79), (3.83), we now introduce

$$\Phi(f_1,f_2,f) := \left(\frac{\eta_j}{|\eta|}(\partial_j u)\Big|_{\partial\Omega}\ ,\ |\eta|^{n-3}\partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big)\Big|_{\partial\Omega}\ ,\ \partial_\nu^- h - T^* f \right), \tag{3.84}$$

where $u, h, \widetilde{h}, H, f_1, f_2, f$ are as in (3.65)-(3.69) *with the convention that* $C_2 = 0$, acting as a linear operator

$$\Phi : L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega). \tag{3.85}$$

Explicitly,

$$\Phi(f_1,f_2,f) = \begin{pmatrix} |\eta|^{-1} S\Big((\tfrac{1}{2}I + K^*)^{-1}(\partial_\nu^- h - T^* f + \mathscr{C}_1 f)\Big) - |\eta|^{-1}h(0) + |\eta|^{-1} f_1 \\ |\eta|^{-1}\Big[\partial_\nu^- h - T^* f\Big] + |\eta|^{-1} f_2 + |\eta|^{-1}\partial_\nu H + (n-2)|\eta|^{-3}\langle \nu, \eta\rangle \widetilde{h} + \mathscr{C}_2 f \\ \partial_\nu^- h - T^* f \end{pmatrix} \tag{3.86}$$

where $\mathscr{C}_1, \mathscr{C}_2$ are as in (3.76) and (3.82), respectively. This suggests the decomposition

$$\Phi = \mathcal{M}_\eta \circ A \circ \Phi_0 + \Phi_1, \tag{3.87}$$

where

$$\mathcal{M}_\eta(g_1,g_2,g) := (|\eta|^{-1} g_1, |\eta|^{-1} g_2, g), \tag{3.88}$$

$$A(g_1,g_2,g) := \Big(S\Big((\tfrac{1}{2}I + K^*)^{-1} g\Big) + g_1\ ,\ g + g_2\ ,\ g \Big), \tag{3.89}$$

$$\Phi_0(f_1,f_2,f) := (f_1, f_2, \partial_\nu^- h - T^* f), \tag{3.90}$$

and

$$\Phi_1(f_1,f_2,f) := \begin{pmatrix} |\eta|^{-1} S\Big((\tfrac{1}{2}I + K^*)^{-1}(\mathscr{C}_1 f)\Big) - |\eta|^{-1} h(0) \\ |\eta|^{-1}\partial_\nu H + (n-2)|\eta|^{-3}\langle \nu, \eta\rangle \widetilde{h} + \mathscr{C}_2 f \\ 0 \end{pmatrix}, \tag{3.91}$$

where $h, \widetilde{h}, H$ are as in (3.66)-(3.67). Note that

$$\mathcal{M}_\eta : h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \longrightarrow h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \tag{3.92}$$

is an isomorphism for every $\frac{n-1}{n} < p < \infty$,

$$A : h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \longrightarrow h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \tag{3.93}$$

is an isomorphism for every $1 - \varepsilon < p < 2 + \varepsilon$, and

$$\Phi_1 : h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \longrightarrow h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \tag{3.94}$$

is a compact operator for every $1 - \varepsilon < p < 2 + \varepsilon$. In addition, thanks to (2.218),

$$\Phi_0 : h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \longrightarrow h^{1,p}(\partial\Omega) \oplus h^p(\partial\Omega) \oplus h^p(\partial\Omega) \tag{3.95}$$

is an isomorphism whenever $2 - \varepsilon < p < 2 + \varepsilon$. In concert, these properties ensure that the operator

$$\begin{aligned} \Phi : L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega) \\ \text{is Fredholm with index zero whenever } 2 + \varepsilon < p < 2 + \varepsilon. \end{aligned} \tag{3.96}$$

To study its null-space, assume that $2 + \varepsilon < p < 2 + \varepsilon$ and consider a triplet $(f_1, f_2, f) \in L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega)$ such that $\Phi(f_1, f_2, f) = 0$. Then, if u is as in (3.65)-(3.69) *for the choice* $C_2 = 0$, it follows that u solves the homogeneous version of (3.64). As a consequence, Proposition 3.4 and the uniqueness results from [43] (for the problem (3.18)) give that necessarily $u \equiv 0$ in Ω. Thus, granted (3.65) and keeping in mind that we take $C_2 = 0$, we have

$$-2\nabla_\eta \mathcal{S} f = \Delta u = 0 \quad \text{in} \;\; \Omega, \tag{3.97}$$

which further implies that

$$0 = u = H + 0 + C_1 + 0 \quad \text{in} \;\; \Omega, \tag{3.98}$$

i.e., $H \equiv -C_1$ in Ω. Hence, $H \equiv 0$ in Ω since $H(0) = 0$ by (3.66). In turn, this forces

$$0 = \nabla_\eta H = \widetilde{h} \quad \text{in} \;\; \Omega, \tag{3.99}$$

i.e., h is constant in Ω. In particular, $\widetilde{h} \equiv 0$ in Ω, and $\partial^+_\nu h = 0$ on $\partial\Omega$. This implies

$$f_2 = \partial^+_\nu h - \partial^-_\nu h = -\partial^-_\nu h. \tag{3.100}$$

Going further, (3.67) and the fact that h is constant in Ω guarantees that $h(0) = h|_{\partial\Omega_+} = (\frac{1}{2}I + K)f_1 - Sf_2$ on $\partial\Omega$. In turn, this gives

$$f_1 = S\Big((\tfrac{1}{2}I + K^*)^{-1}f_2\Big) + h(0) \quad \text{on} \quad \partial\Omega. \tag{3.101}$$

On the other hand, $\nabla_\eta \mathcal{S}f \equiv 0$ in Ω gives, on account of (2.33),

$$\mathcal{S}f \equiv constant \quad \text{in} \quad \Omega. \tag{3.102}$$

Recall the function f_o from (2.134) and the fact that the operator (2.135) is an isomorphism whenever $1 < p < 2 + \varepsilon$. In light of this and of (3.102), we then obtain

$$f \in \langle f_o \rangle := \{Cf_o : C \in \mathbb{R}\}. \tag{3.103}$$

Finally, after recalling (3.100), an inspection of the third line of Φ in (3.86) gives that

$$f_2 = -T^* f. \tag{3.104}$$

In summary, the above analysis shows that $\operatorname{Ker}\Phi$ is the two-dimensional space

$$\begin{aligned}&\{(f_1, f_2, f) \in L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega) : \Phi(f_1, f_2, f) = 0\}\\ &\quad = \Big\{\Big(C_1\, S(\tfrac{1}{2}I + K^*)^{-1}T^* f_o + C_2\,,\, -C_1\, T^* f_o\,,\, C_1\, f_o\Big) : C_1, C_2 \in \mathbb{R}\Big\}\end{aligned} \tag{3.105}$$

whenever $2 - \varepsilon < p < 2 + \varepsilon$. By relying on this, (3.96) and Fredholm theory, one can then easily show that the natural operator induced by Φ at the level

$$\widetilde{\Phi} : \Big[L^p_1(\partial\Omega)/\mathbb{R}\Big] \oplus L^p(\partial\Omega) \oplus L^p_0(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \oplus L^p_\eta(\partial\Omega) \oplus L^p(\partial\Omega) \tag{3.106}$$

is one-to-one with closed range of codimension one, whenever $2 + \varepsilon < p < 2 + \varepsilon$. Above, $L^p_\eta(\partial\Omega)$ and $L^p_0(\partial\Omega)$ are as in (3.62) and (3.63), respectively.

We now claim that

$$\begin{aligned}&\widehat{\Phi} : \Big[L^p_1(\partial\Omega)/\mathbb{R}\Big] \oplus L^p(\partial\Omega) \oplus L^p_0(\partial\Omega) \longrightarrow \Big[L^p_1(\partial\Omega)/\langle\, |\eta|\,\rangle\Big] \oplus L^p_\eta(\partial\Omega) \oplus L^p(\partial\Omega)\\ &\qquad \text{is an isomorphism whenever } 2 + \varepsilon < p < 2 + \varepsilon,\end{aligned} \tag{3.107}$$

where $\widehat{\Phi}$ is the operator naturally induced by Φ in this context. Indeed, from what we have just proved about the operator (3.106), it follows that $\widehat{\Phi}$ in (3.107) is Fredholm with index zero. Thus, it suffices to show that this is also one-to-one. To this end, assume that the triplet $(f_1, f_2, f) \in L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p_0(\partial\Omega)$ is such that $\Phi(f_1, f_2, f) = (C|\eta|, 0, 0)$ and define u as in (3.65)-(3.69) with $C_2 = 0$. Then, as is visible from (3.65),

$$\Delta u(0) = 0 \tag{3.108}$$

and u solves

$$\begin{cases} \Delta^2 u = 0 \ \text{ in } \ \Omega, \\ \dfrac{\eta_j}{|\eta|}(\partial_j u)\Big|_{\partial\Omega} = C|\eta| \text{ on } \partial\Omega, \\ |\eta|^{n-3}\partial_{\tau_{jk}}\Big(\dfrac{\eta_k}{|\eta|^{n-2}}\partial_j u\Big)\Big|_{\partial\Omega} = 0 \text{ on } \partial\Omega, \\ M(\nabla\nabla u) \in L^p(\partial\Omega), \quad u(0) = 0. \end{cases} \tag{3.109}$$

In order to continue, observe that

$$\frac{\eta_j}{|\eta|}(\partial_j|\eta|^2) = 2|\eta| \quad \text{and} \quad |\eta|^{n-3}\partial_{\tau_{jk}}\Big(\frac{\eta_k}{|\eta|^{n-2}}\partial_j|\eta|^2\Big)\Big|_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.110}$$

The second equality follows by noting that $(\nu_j\partial_k - \nu_k\partial_j)(x_j x_k|X|^{2-n}) = 0$, since the expressions in the first and second set of parentheses are, respectively, antisymmetric and symmetric in j, k. Thus, if we set

$$v(X) := u(X) - \frac{C}{2}|X|^2, \qquad X \in \Omega, \tag{3.111}$$

it follows that v is a solution of the homogeneous version of (3.64). Much as before, this implies that $v \equiv 0$ in Ω, i.e.,

$$u(X) = \frac{C}{2}|X|^2, \qquad X \in \Omega. \tag{3.112}$$

In turn, this and (3.108) give that

$$0 = \Delta u(0) = n\,C, \tag{3.113}$$

i.e., $C = 0$. Consequently, $u \equiv 0$ in Ω, by (3.112). With this in hand, the same analysis as in the first half of the current proof shows that (f_1, f_2, f) belongs to the space in the right-hand side of (3.105). Next, since f integrates to zero on $\partial\Omega$ and f_o does not, we may deduce from this that $f = f_2 = 0$ and f_1 is a constant on $\partial\Omega$. Hence, with $[f_1]$ denoting the class of f_1

(in this case, modulo constants), $([f_1], f_2, f)$ is zero in $\Big[L^p_1(\partial\Omega)/\mathbb{R}\Big] \oplus L^p(\partial\Omega) \oplus L^p_0(\partial\Omega)$. This shows that $\widehat{\Phi}$ in (3.107) is one-to-one and, hence, an isomorphism.

Having established (3.107), the representation formula (3.65) for the solution u of (3.64) is justified as follows. Start by taking

$$([f_1], f_2, f) := \widehat{\Phi}^{-1}([g_0], g_1, 0), \tag{3.114}$$

then consider the function H as in (3.66)-(3.67), and the constant C_1 as in (3.69). The function defined by $v(X) := H(X) - 2\Gamma(\nabla_\eta \mathcal{S} f)(X) - C_1$, for $X \in \Omega$, then solves

$$\begin{cases} \Delta^2 v = 0 \ \text{ in } \Omega, \\ \dfrac{\eta_j}{|\eta|}(\partial_j v)\Big|_{\partial\Omega} = g_0 + C|\eta| \text{ on } \partial\Omega, \\ |\eta|^{n-3}\partial_{\tau_{jk}}\Big(\dfrac{\eta_k}{|\eta|^{n-2}}\partial_j v\Big)\Big|_{\partial\Omega} = g_2 \text{ on } \partial\Omega, \\ M(\nabla\nabla v) \in L^p(\partial\Omega), \quad v(0) = 0, \end{cases} \tag{3.115}$$

for some constant $C \in \mathbb{R}$. With this in hand and recalling (3.110), it follows that u as in (3.65) solve (3.64) for the choice $C_2 := C/2$.

Uniqueness for (3.64) follows from Proposition 3.4 and the corresponding uniqueness for (3.18) (when $M(\nabla\nabla u) \in L^p(\partial\Omega)$ with $|p-2| < \varepsilon$) from [43].

Finally, consider (3.70)-(3.71). Recall that, when $2 - \varepsilon < p < 2 + \varepsilon$, u is given by the integral representation formula (3.65) with $f_1 \in L^p_1(\partial\Omega)$ and $f_2, f \in L^p(\partial\Omega)$. The claim we make is that $u \in B^{p,p\vee 2}_{2+1/p}(\Omega)$, plus estimates. Given that $u \in \operatorname{Ker}\Delta^2$, the estimates (3.184)-(3.185) will then follow from this claim and Theorem 4.16.

To prove that $u \in B^{p,p\vee 2}_{2+1/p}(\Omega)$, we first note that the piece corresponding to the Newtonian potential is of the right order, by (2.146) and Proposition 4.23. To see that H in (3.66)-(3.67) is also in $B^{p,p\vee 2}_{2+1/p}(\Omega)$, we write $\nabla_\eta H = \widetilde{h} \in B^{p,p\vee 2}_{1+1/p}(\Omega)$, by Proposition 4.23 (here, (2.120) and (4.93) are also used). With this in hand, the desired conclusion follows from Proposition 4.17. This finishes the proof of the theorem. □

Let $\pi_j(x_1, x_2, x_3) := x_j$, $1 \le j \le 3$, be the canonical projection onto the j-th component.

Lemma 3.7. *For $2 - \varepsilon < p < 2 + \varepsilon$, define the bounded, linear assignment*

$$\Psi : L^p_\eta(\partial\Omega) \longrightarrow L^p_0(\partial\Omega), \qquad \Psi g := \pi_3(\widehat{\Phi}^{-1}([0], g, 0)). \tag{3.116}$$

Then the above operator is well-defined, linear, bounded and one-to-one.

Proof. Fix $p \in (2-\varepsilon, 2+\varepsilon)$ and suppose that $\Psi g = 0$ for some $g \in L^p_\eta(\partial\Omega)$. Thus, if we set $([f_1], f_2, f) := \widehat{\Phi}^{-1}([0], g, 0)$, our assumptions give that $f = 0$. For the functions f_1, f_2, f, consider now $u, h, \widetilde{h}, H$ as in (3.65)-(3.69) with $C_2 = 0$. Then, from the format of the third line of Φ in (3.86), we obtain $\partial_\nu^- h = T^* f = 0$. By the uniqueness in the exterior Neumann problem and the decay of the function h at infinity, this forces

$$h \equiv 0 \quad \text{in} \quad \Omega_-. \tag{3.117}$$

Also, recalling (3.86), we have

$$[0] = \pi_1([0], g, 0) = \pi_1(\widehat{\Phi}([f_1], f_2, 0)) = [\, |\eta|^{-1}(f_1 - h(0))\,]. \tag{3.118}$$

This means that there exists $C \in \mathbb{R}$ such that $|\eta|^{-1}(f_1 - h(0)) = C|\eta|$ on $\partial\Omega$, i.e.,

$$f_1 = C|\eta|^2 + h(0) \quad \text{on} \quad \partial\Omega. \tag{3.119}$$

This and (3.117) then imply that $h|_{\partial\Omega_+} = h|_{\partial\Omega_-} + f_1 = C|\eta|^2 + h(0)$ or, equivalently,

$$\widetilde{h}\Big|_{\partial\Omega_+} = C|\eta|^2. \tag{3.120}$$

Since $\widetilde{h}$ is harmonic in Ω and vanishes at the origin, if $C \neq 0$ this would imply $|\eta|^2 \in L^p_\omega(\partial\Omega)$, contradicting (2.138). Thus, necessarily $C = 0$, which then entails $\widetilde{h}|_{\partial\Omega_+} = 0$. Hence, by the uniqueness in the Dirichlet problem,

$$h \equiv \textit{constant} \quad \text{in} \quad \Omega_+, \tag{3.121}$$

which forces $\widetilde{h} \equiv 0$ in Ω and, further, $H \equiv 0$ in Ω. In turn, this and the format of the second line in (3.86) allow us to write

$$g = \pi_2([0], g, 0) = \pi_2(\widehat{\Phi}([f_1], f_2, 0)) = |\eta|^{-1} f_2, \tag{3.122}$$

i.e. $g = |\eta|^{-1} f_2$. However, since $\mathcal{D}1 = 0$ in Ω_-, we may write $0 = h|_{\Omega_-} = -\mathcal{S} f_2$, yielding $f_2 = 0$. Thus, $g = 0$, proving that the mapping (3.116) is one-to-one. $\square$

Lemma 3.8. *For $2 - \varepsilon < p < 2 + \varepsilon$, the operator Ψ in (3.116) is Fredholm, of index zero.*

Proof. For an arbitrary $g \in L^p_\eta(\partial\Omega)$ consider $f := \Psi g$ and set $[f_1] := \pi_1(\widehat{\Phi}^{-1}([0], g, 0))$, $f_2 := \pi_2(\widehat{\Phi}^{-1}([0], g, 0))$. In particular, $f_1 \in L^p_1(\partial\Omega)$, $f_2 \in L^p(\partial\Omega)$ satisfy $\Phi(f_1, f_2, f) = (0, g, 0)$. Referring to (3.86), this gives

$$T^* f = \partial_\nu^- h = \partial_\nu \mathcal{D} f_1 - (\tfrac{1}{2} I + K^*) f_2 \tag{3.123}$$

as well as

$$f_1 + \mathscr{C}(f) = 0, \qquad f_2 + \mathscr{C}(f_1, f_2, f) = g, \tag{3.124}$$

where we have denoted by $\mathscr{C}$ generic compact operators. Since $[f_1]$ and f_2 depend continuously on g, we may rewrite (3.124) as

$$f_1 = \mathscr{C}(g), \qquad f_2 = g + \mathscr{C}(g). \tag{3.125}$$

Returning with these in (3.123) we obtain

$$T^* f = -(\tfrac{1}{2}I + K^*)g + \mathscr{C}(g) \tag{3.126}$$

so that

$$\Psi g = f = -(T^*)^{-1}(\tfrac{1}{2}I + K^*)g + \mathscr{C}(g). \tag{3.127}$$

That is,

$$\Psi = -(T^*)^{-1}(\tfrac{1}{2}I + K^*) + \mathscr{C}. \tag{3.128}$$

Observing that $(T^*)^{-1}(\frac{1}{2}I + K^*) + \mathscr{C} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is a Fredholm operator, of index zero, the desired conclusion readily follows. □

Proposition 3.9. *If Ω is as in (2.4), then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the operator Ψ in (3.116) is invertible for each $p \in (2 - \varepsilon, 2 + \varepsilon)$.*

Proof. This is an immediate consequence of Lemma 3.7 and Lemma 3.8. □

3.5 The nonstandard version with atomic data

The following is Theorem 9.6 on p. 965 in [43]; see also Theorem 1.2 on p. 619 in [45].

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary. Then, whenever*

$$f_0, f_1, f_2, f_3 \in h^{1,1}_{at}(\partial\Omega), \tag{3.129}$$

the problem (3.18)-(3.19) has a unique solution with $M(\nabla\nabla u) \in L^1(\partial\Omega)$. In addition,

$$\|M(\nabla\nabla u)\|_{L^1(\partial\Omega)} \leq C \sum_{j,k,r=1}^{3} \|\partial_{\tau_{jk}} f_r\|_{H^1_{at}(\partial\Omega)}, \tag{3.130}$$

where the constant C depends only on the Lipschitz character of Ω.

The goal is to use the results from § 3.4 in order to convert the above theorem into a well-posedness result from the problem (3.28) with atomic data.

Proposition 3.11. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists a constant $C > 0$, depending only on the Lipschitz character of Ω, with the following significance. Let u be a solution (in the L^2 sense) of the problem*

$$\begin{cases} \Delta^2 u = 0 \quad in \ \Omega, \\ \eta_j \partial_j u = 0 \quad on \ \partial\Omega, \\ \partial_{\tau_{jk}} \Big(\frac{\eta_k}{|\eta|} \partial_j u \Big) = a \quad on \ \partial\Omega, \end{cases} \tag{3.131}$$

where the datum $a : \partial\Omega \to \mathbb{R}$ satisfies (for some point $X_o \in \partial\Omega$ and some small $r > 0$),

$$\operatorname{supp} a \subseteq \partial\Omega \cap B(X_o, r), \qquad \|a\|_{L^\infty(\partial\Omega)} \leq r^{-2}, \qquad \int_{\partial\Omega} a \, d\sigma = 0. \tag{3.132}$$

Then

$$\|M(\nabla\nabla u)\|_{L^1(\partial\Omega)} \leq C. \tag{3.133}$$

Proof. Proposition 3.5 with $n = 3$ and $g_1(X) := a(X)$ for $X \in \partial\Omega$, gives that (modulo an additive constant) u solves the problem (3.18)-(3.19) with

$$f_0 := u_o\Big|_{\partial\Omega}, \qquad f_j := (\partial_j u_o)\Big|_{\partial\Omega}, \qquad 1 \leq j \leq 3, \tag{3.134}$$

where $u_o(X) := w(X/|X|)$, $X \in \mathbb{R}^3 \setminus \{0\}$, and w solves

$$\Delta_{S^2} w(\omega) = -a(\varphi(\omega)\omega)\varphi(\omega)\sqrt{|(\nabla_{tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}, \qquad \omega \in S^2. \tag{3.135}$$

Observe that $a(\varphi(\omega)\omega)\varphi(\omega)\sqrt{|(\nabla_{tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}$ is, up to a fixed multiplicative constant, an atom in $H^1_{at}(S^2)$, so elliptic regularity (on the unit sphere) implies that

$$\|\nabla_{tan} w\|_{h^{1,1}_{at}(S^2)} \leq C, \tag{3.136}$$

for some C depending only on the Lipschitz constant of φ. Since for $X = \varphi(\omega)\omega \in \partial\Omega$, $\omega \in S^2$,

$$f_j(X) = (\partial_j u_o)(X) = \varphi(\omega)^{-1}(\nabla_{tan} w)_j(\omega), \qquad 1 \leq j \leq 3, \tag{3.137}$$

and since the mapping $h^{1,1}_{at}(S^2) \ni v \mapsto \widehat{v} \in h^{1,1}_{at}(\partial\Omega)$ with $\widehat{v}(X) := v(X/|X|)$ is bounded (as is easily seen at the atomic level), it follows from (3.136) that

$$\|f_j\|_{h^{1,1}_{at}(\partial\Omega)} \leq C, \qquad 1 \leq j \leq 3. \tag{3.138}$$

With this in hand, (3.133) follows from Theorem 3.10. □

Having established Proposition 3.11, we wish to use this atomic estimate in order to prove the boundedness and, later, invertibility, of certain boundary operators on Hardy spaces. For the time being, we aim at proving the following.

Proposition 3.12. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain which is star-like with respect to the origin. Then Ψ in (3.116) extends to a linear, bounded operator*

$$\Psi : H^1_{at}(\partial\Omega) \longrightarrow H^1_{at}(\partial\Omega). \tag{3.139}$$

Proof. Consider the solution u of (3.64) (in the L^2 sense) with data $g_0 \equiv 0$ and $g_1 := a$, where a is as in (3.132). From the discussion in § 3.4, we have

$$u = H - 2\Gamma(\nabla_\eta \mathcal{S}(\Psi a)) + C_1 + C_2|\eta|^2 \quad \text{in } \Omega, \tag{3.140}$$

where

$$C_1 = \nabla_\eta \mathcal{S}(\Psi a)(0), \qquad C_2 = \Delta u(0)/6. \tag{3.141}$$

In particular, $\Delta u(X) = \nabla_\eta \mathcal{S}(\Psi a)(X) + \Delta u(0)$, for $X \in \Omega$. By virtue of Lemma 2.12 and Proposition 3.11, this implies

$$\begin{aligned} \|M(\nabla \mathcal{S}(\Psi a))\|_{L^1(\partial\Omega)} &\leq C\|M(\nabla_\eta \mathcal{S}(\Psi a))\|_{L^1(\partial\Omega)} \\ &\leq C\|M(\Delta u)\|_{L^1(\partial\Omega)} + C\,|\Delta u(0)| \\ &\leq C\|M(\Delta u)\|_{L^1(\partial\Omega)} \leq C, \end{aligned} \tag{3.142}$$

for some finite $C = C(\Omega) > 0$, independent of the atom a. Next, recall that

$$\Delta v = 0 \text{ in } \Omega \implies \|\partial_\nu v\|_{H^1_{at}(\partial\Omega)} \leq C\|M(\nabla v)\|_{L^1(\partial\Omega)}, \tag{3.143}$$

where $C = C(\Omega) > 0$ is independent of v. Cf. [11] for a proof. Utilizing this for the choice $v := \mathcal{S}(\Psi a)$ gives, after a reference to (3.142), that

$$\|(-\tfrac{1}{2}I + K^*)(\Psi a)\|_{H^1_{at}(\partial\Omega)} \leq C, \tag{3.144}$$

for every atom a in $H^1_{at}(\partial\Omega)$. Since $\Psi a \in L^2_0(\partial\Omega) \hookrightarrow H^1_{at}(\partial\Omega)$ and

$$-\tfrac{1}{2}I + K^* : H^1_{at}(\partial\Omega) \longrightarrow H^1_{at}(\partial\Omega) \tag{3.145}$$

is an isomorphism (cf. [11]), we finally arrive at the conclusion that there exists a finite constant $C = C(\Omega) > 0$ such that

$$\|\Psi a\|_{H^1_{at}(\partial\Omega)} \leq C, \tag{3.146}$$

for every atom a in $H^1_{at}(\partial\Omega)$. Granted this, and given Lemma 3.7, it follows that the operator (3.139) is bounded. □

The above end-point boundedness result further entails the following.

Proposition 3.13. *If $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain which is star-like with respect to the origin then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the property that the operator Ψ in (3.116) is well-defined and bounded whenever $1 < p < 2 + \varepsilon$.*

Proof. This follows by interpolating (by the complex method) the results in Lemma 3.7 and Proposition 3.12. □

Next, assuming that Ω is as in (2.4) and $|p - 2| < \varepsilon$, consider the operator

$$\begin{aligned} &\Theta : L^p_\eta(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \Theta g := \partial^-_\nu(\mathcal{D}f_1 - \mathcal{S}f_2), \\ &\text{where } \ [f_1] := \pi_1(\widehat{\Phi}^{-1}([0], g, 0)), \quad f_2 := \pi_2(\widehat{\Phi}^{-1}([0], g, 0)). \end{aligned} \tag{3.147}$$

Unraveling definitions we see that, whenever $2 - \varepsilon < p < 2 + \varepsilon$,

$$T^* \circ \Psi = \Theta, \qquad \text{as operators mapping } L^p_\eta(\partial\Omega) \text{ into } L^p(\partial\Omega). \tag{3.148}$$

Next, we desire an alternative description of Θ in (3.147) which emphasizes Ψ in place of $\widehat{\Phi}^{-1}$. With this goal in mind, fix an arbitrary $g \in L^p_\eta(\partial\Omega)$ and set $([f_1], f_2, f) := \widehat{\Phi}^{-1}([0], g, 0)$, so that

$$\Phi(f_1, f_2, f) = (C|\eta|, g, 0), \qquad \text{for some } \ C \in \mathbb{R}. \tag{3.149}$$

Consequently, if $h, \widetilde{h}, H$ are as in (3.66)-(3.67), then

$$f = \Psi g \tag{3.150}$$

by (3.116) and, by (3.86), (3.65),

$$C=-\frac{1}{n}\,\Delta u(0), \tag{3.151}$$

where u is the unique solution of (3.64) with data $g_0:=0$ and $g_1:=g$. In particular, the assignment $\mathcal{B}:g\mapsto C$ is linear and bounded, by Theorem 3.6, when acting from $L^p_\eta(\partial\Omega)$ with $2-\varepsilon<p<2+\varepsilon$. If $n=3$, the same is true when $\mathcal{B}$ acts from $H^1_{at}(\partial\Omega)$, by Proposition 3.11. By interpolation, it follows that $\mathcal{B}$ acts boundedly from $L^p_0(\partial\Omega)$ when $n=3$, provided $1<p<2+\varepsilon$.

Going further, (3.149) implies $T^*f=\partial_\nu^- h$ and

$$S\Big((\tfrac12 I+K^*)^{-1}(\mathscr{C}_1 f)\Big)-h(0)+f_1=|\eta|^2\mathcal{B}g, \tag{3.152}$$

$$f_2+(n-2)|\eta|^{-2}\langle\nu,\eta\rangle\widetilde{h}+\mathscr{C}_2 f=|\eta|g, \tag{3.153}$$

by (3.86). Let us schematically re-write (3.152)-(3.152) in the form

$$(I+\mathscr{C})\begin{pmatrix}\xi\\ f_2\end{pmatrix}=A\begin{pmatrix}\Psi g\\ g\end{pmatrix}+\begin{pmatrix}0\\ |\eta|g\end{pmatrix} \tag{3.154}$$

where $\xi:=f_1-h(0)$ and

$$\mathscr{C}:=(n-2)|\eta|^2\langle\nu,\eta\rangle\begin{pmatrix}0 & 0\\ \frac12 I+K & -S\end{pmatrix},\qquad A:=\begin{pmatrix}A_1 & A_3\\ A_2 & 0\end{pmatrix}, \tag{3.155}$$

with

$$A_1:=-S(\tfrac12 I+K^*)^{-1}\mathscr{C}_1,\qquad A_2:=-\mathscr{C}_2,\qquad A_3:=|\eta|^2\mathcal{B}, \tag{3.156}$$

where $\mathcal{B}$, $\mathscr{C}_1$ and $\mathscr{C}_2$ are as before. Then

$$\mathscr{C}:h^{1,p}(\partial\Omega)\oplus h^p(\partial\Omega)\longrightarrow h^{1,p}(\partial\Omega)\oplus h^p(\partial\Omega),\qquad \tfrac{n-1}{n}<p<\infty, \tag{3.157}$$

is compact and *smoothing*, in the sense that, when applied to functions, the output has a better integrability exponent than the input. Let us also observe that, by Lemma 2.8, the operator

$$I+\mathscr{C}:h^{1,p}(\partial\Omega)\oplus h^p(\partial\Omega)\longrightarrow h^{1,p}(\partial\Omega)\oplus h^p(\partial\Omega) \tag{3.158}$$

is invertible for $\frac{n-1}{n}<p<\infty$.

Specializing this discussion to the case when $n=3$, we have

$$\begin{pmatrix} f_1 - h(0) \\ f_2 \end{pmatrix} = (I+\mathscr{C})^{-1}\left[A\begin{pmatrix}\Psi g \\ g\end{pmatrix} + \begin{pmatrix} 0 \\ |\eta| g\end{pmatrix}\right]$$
$$= (I-(I+\mathscr{C})^{-1}\mathscr{C})\left[A\begin{pmatrix}\Psi g \\ g\end{pmatrix} + \begin{pmatrix} 0 \\ |\eta| g\end{pmatrix}\right] \tag{3.159}$$

which we rewrite as

$$\begin{pmatrix} f_1 - h(0) \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ |\eta| g\end{pmatrix} + \begin{pmatrix} \mathscr{C}' g \\ \mathscr{C}'' g\end{pmatrix} \tag{3.160}$$

for some smoothing compact operators $\mathscr{C}'$, $\mathscr{C}''$ (here Proposition 3.13 is used). This finally gives

$$\Theta g = \partial_\nu^-(\mathcal{D} f_1 - \mathcal{S} f_2) = \partial_\nu^-(\mathcal{D}(f_1 - h(0)) - \mathcal{S} f_2)$$
$$= \Big(\partial_\nu \mathcal{D}\,,\, -(\tfrac{1}{2}I + K^*)\Big)\begin{pmatrix} f_1 - h(0) \\ f_2 \end{pmatrix} = -(\tfrac{1}{2}I + K^*)(|\eta| g) + \widetilde{\mathscr{C}} g, \tag{3.161}$$

for some operator $\widetilde{\mathscr{C}}$ which is smoothing and compact on $h^p(\partial\Omega)$, for $1 \leq p < 2+\varepsilon$.

Proposition 3.14. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain which is star-like with respect to the origin. There exists $\varepsilon > 0$ with the property that (3.147) extends to an isomorphic embedding (i.e., as a one-to-one operator with closed range), with finite dimensional cokernel both in the context*

$$\Theta : H^1_{at}(\partial\Omega) \longrightarrow h^1_{at}(\partial\Omega), \tag{3.162}$$

as well as in

$$\Theta : L^p_0(\partial\Omega) \longrightarrow L^p(\partial\Omega), \qquad 1 < p < 2+\varepsilon. \tag{3.163}$$

Proof. For starters, we claim that in each instance the operator in question is Fredholm. Indeed, this is an immediate consequence of the representation of Θ obtained in (3.161), since both multiplication by $|\eta|$ and $\frac{1}{2}I + K^*$ are Fredholm operators on $H^1_{at}(\partial\Omega)$ and $L^p(\partial\Omega)$ provided $1 < p < 2+\varepsilon$. This ensures that the range is closed and of finite codimension. As for being one-to-one, since $\frac{1}{2}I + K^*$ is invertible on $L^p(\partial\Omega)$ if $1 < p < 2+\varepsilon$, and since $\widetilde{\mathscr{C}}$ in (3.161) is smoothing, the null-spaces of Θ in (3.162) and (3.163) coincide with that of Θ in (3.147) when $p = 2$. In this latter case, however, (3.148), Lemma 3.7 and Theorem 2.19 ensure that $\operatorname{Ker}\Theta = \{0\}$. $\square$

With Proposition 3.14 in hand, we now return to the study of the functional analytic properties of the operator Ψ.

Proposition 3.15. *Suppose that* $\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz domain which is star-like with respect to the origin. There exists* $\varepsilon > 0$ *with the property that*

$$\Psi : H^1_{at}(\partial\Omega) \longrightarrow H^1_{at}(\partial\Omega), \tag{3.164}$$

$$\Psi : L^p_0(\partial\Omega) \longrightarrow L^p_0(\partial\Omega), \quad 1 < p < 2 + \varepsilon, \tag{3.165}$$

are isomorphisms.

Proof. Since $T^* \circ \Psi = \Theta$ with Θ an isomorphic embedding and T^* bounded, it follows that Ψ has closed range both in (3.164) and (3.165). Since, by Lemma 3.8, Ψ has dense range in both instances, it follows that Ψ in (3.164)-(3.165) is actually onto. Finally, since $T^* \circ \Psi = \Theta$ and Θ is one-to-one, we may also deduce that Ψ is one-to-one. □

We are now in a position to establish the Fredholmness of T^*, which is really the main operator of interest for us, in a more general setting than that considered in Theorem 2.19.

Proposition 3.16. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded Lipschitz domain which is star-like with respect to the origin. There exists* $\varepsilon > 0$ *with the property that*

$$T^* : h^1_{at}(\partial\Omega) \longrightarrow h^1_{at}(\partial\Omega), \tag{3.166}$$

$$T^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < 2 + \varepsilon, \tag{3.167}$$

are Fredholm operators.

Proof. Note that

$$T^* = \Theta \circ \Psi^{-1} : H^1_{at}(\partial\Omega) \longrightarrow h^1_{at}(\partial\Omega) \tag{3.168}$$

is Fredholm by Proposition 3.15 and Proposition 3.14. Since the difference between the context in (3.166) and that in (3.168) is a space finite dimension, it readily follows that T^* in (3.166) is Fredholm. The reasoning for (3.167) is similar. □

Making use of stability results, Proposition 3.16 can be further sharpened as follows.

Theorem 3.17. *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz domain which is star-like with respect to the origin. There exists* $\varepsilon > 0$ *with the property that*

$$T^* : h^p_{at}(\partial\Omega) \longrightarrow h^p_{at}(\partial\Omega), \quad 1 - \varepsilon < p \leq 1, \tag{3.169}$$

$$T^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < 2 + \varepsilon, \tag{3.170}$$

are invertible operators.

Proof. Combining Theorem 2.19 and Proposition 3.16 we obtain that the operators (3.166)-(3.167) are onto if $1 \leq p < 2+\varepsilon$. As proved in [23], the property of being onto is stable on complex interpolation scales of quasi-Banach spaces. Consequently, there exists $\varepsilon > 0$ such that the operator (3.169) is onto. With this in hand, the fact that T^* in (3.169)-(3.170) is an isomorphism follows from Theorem 2.19 and Theorem 2.10 in [23]. □

Corollary 3.18. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain which is star-like with respect to the origin. There exists a small number $\varepsilon > 0$ such that*

$$T : C^\alpha(\partial\Omega) \longrightarrow C^\alpha(\partial\Omega) \quad \textit{if } \ 0 < \alpha < \varepsilon, \tag{3.171}$$

$$T : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{if } \ 2-\varepsilon < p < \infty, \tag{3.172}$$

$$T : \mathrm{bmo}\,(\partial\Omega) \longrightarrow \mathrm{bmo}\,(\partial\Omega), \tag{3.173}$$

$$T : \mathrm{vmo}\,(\partial\Omega) \longrightarrow \mathrm{vmo}\,(\partial\Omega), \tag{3.174}$$

isomorphically.

Proof. The fact that the operators (3.171)-(3.173) are isomorphisms is a direct consequence of Theorem 3.17 and duality (cf. the discussion in § 2.3). As for (3.174), first observe that, in this context, T is well-defined, linear and bounded by (2.62), the boundedness of T on $\mathrm{bmo}\,(\partial\Omega)$ and the fact that T maps $C^\alpha(\partial\Omega)$ into itself. Furthermore, by (3.171) and (3.173), T in (3.174) is one-to-one, with dense range. By (2.60), the dual of the latter operator is T^* acting on $h^1_{at}(\partial\Omega)$, thus an isomorphism by Theorem 3.17. Then, Banach's Closed Range Theorem ensures that T in (3.174) has closed range as well. Hence, all in all, T in (3.174) is also an isomorphism. □

The following is an extension of the claim in (3.107).

Theorem 3.19. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists $\varepsilon > 0$ such that the operator $\widehat{\Phi}$ in (3.107) extends isomorphically as*

$$\begin{aligned}\widehat{\Phi} : \Big[h^{1,p}_{at}(\partial\Omega)/\mathbb{R}\Big] \oplus h^p_{at}(\partial\Omega) \oplus H^p_{at}(\partial\Omega) \qquad\qquad\qquad\qquad & \qquad (3.175)\\ \longrightarrow \Big[h^{1,p}_{at}(\partial\Omega)/\langle\,|\eta|\,\rangle\Big] \oplus H^p_{at}(\partial\Omega) \oplus h^p_{at}(\partial\Omega) \quad \textit{if } \ 1-\varepsilon < p \leq 1, &\end{aligned}$$

$$\begin{aligned}\widehat{\Phi} : \Big[L^p_1(\partial\Omega)/\mathbb{R}\Big] \oplus L^p(\partial\Omega) \oplus L^p_0(\partial\Omega) \qquad\qquad\qquad\qquad & \\ \longrightarrow \Big[L^p_1(\partial\Omega)/\langle\,|\eta|\,\rangle\Big] \oplus L^p_0(\partial\Omega) \oplus L^p(\partial\Omega) \quad \textit{if } \ 1 < p < 2+\varepsilon. & \qquad (3.176)\end{aligned}$$

Proof. The operators

$$\Phi : h^{1,p}_{at}(\partial\Omega) \oplus h^p_{at}(\partial\Omega) \oplus h^p_{at}(\partial\Omega) \longrightarrow h^{1,p}_{at}(\partial\Omega) \oplus h^p_{at}(\partial\Omega) \oplus h^p_{at}(\partial\Omega) \quad \text{if } \tfrac{2}{3} - \varepsilon < p \leq 1, \tag{3.177}$$

$$\Phi : L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \oplus L^p(\partial\Omega) \quad \text{if } 1 < p < \infty, \tag{3.178}$$

are well-defined, linear and bounded. Having established Theorem 3.17, the same argument as before (based on the representation (3.87)) shows that there exists $\varepsilon > 0$ such that Φ in (3.177) is Fredholm with index zero if $1 - \varepsilon < p \leq 1$, whereas Φ in (3.178) is Fredholm with index zero if $1 < p < 2 + \varepsilon$. This, in turn, ensures that (3.175)-(3.176) are also Fredholm operators with index zero. Since they are known to actually be invertible if p is near 2 (cf. (3.107)), the claim in the statement of the theorem follows from this and elementary functional analysis. □

3.6 The Dirichlet and regularity problems in three dimensions

The results in § 3.1 and § 3.4, established in all space dimensions $n \geq 3$, have been established only for p near 2. The goal of this section is to sharpen these results in the case when the ambient space is three-dimensional.

Theorem 3.20. *For every bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ which is star-like with respect to the origin there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that if $2 - \varepsilon < p < \infty$ then the boundary value problem (3.1)-(3.2) is uniquely solvable for any $g_0 \in L^p_1(\partial\Omega)$ and $g_1 \in L^p(\partial\Omega)$. Furthermore, the solution u can be represented as in (3.3), where f is as in (3.4), and (3.6) is satisfied for $2 \leq p < \infty$.*

Proof. Granted Corollary 3.18, existence and uniqueness can be established as in the proof of Theorem 3.1. Finally, the fact that (3.6) holds for the larger range $2 \leq p < \infty$ is a consequence of the invertibility of T in (3.172) and the same reasoning as before. □

To proceed, we record a trace result which shows that the very formulation of the regularity problem with nontangential maximal function estimates remains meaningful for data selected from Hardy spaces with $p < 1$.

Proposition 3.21. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $u \in C^1(\Omega)$ is such that $M(\nabla u) \in L^p(\partial\Omega)$ for some $p \in (\frac{n-1}{n}, 1]$. Then u has a nontangential limit at almost every boundary point on $\partial\Omega$ and, for any $q \in (0, \infty]$,*

$$u\Big|_{\partial\Omega} \in h^{1,p}_{at}(\partial\Omega) \quad \textit{and} \quad \|u|_{\partial\Omega}\|_{h^{1,p}_{at}(\partial\Omega)} \leq C\|M(\nabla u)\|_{L^p(\partial\Omega)} + C\|u\|_{L^q(\mathcal{O})}, \tag{3.179}$$

for some relatively compact subset $\mathcal{O} \subset \Omega$. In particular,

$$\|u|_{\partial\Omega}\|_{h^{1,p}_{at}(\partial\Omega)} \leq C\|M(\nabla u)\|_{L^p(\partial\Omega)} + C\|Mu\|_{L^p(\partial\Omega)}. \tag{3.180}$$

Furthermore,

$$\sum_{j,k=1}^{n} \|\partial_{\tau_{jk}}(u|_{\partial\Omega})\|_{H^p_{at}(\partial\Omega)} \leq C\|M(\nabla u)\|_{L^p(\partial\Omega)}. \tag{3.181}$$

See [36] for a proof. Once again, this result ensures that the formulation of (3.18)-(3.19) is meaningful for all indices $p \in (\frac{n-1}{n}, \infty)$.

Theorem 3.22. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the property that if*

$$f_0, f_1, f_2, f_3 \in h^{1,p}(\partial\Omega), \qquad 1 - \varepsilon < p < 2 + \varepsilon, \tag{3.182}$$

then the problem (3.18)-(3.19) has a unique solution with $M(\nabla\nabla u) \in L^p(\partial\Omega)$. Moreover,

$$\|M(\nabla\nabla u)\|_{L^p(\partial\Omega)} \leq C \sum_{j,k,r=1}^{3} \|\partial_{\tau_{jk}} f_r\|_{h^p(\partial\Omega)}, \tag{3.183}$$

for some finite $C = C(\Omega) > 0$. Furthermore,

$$\|u\|_{B^{p,2}_{2+\frac{1}{p}}(\Omega)} \leq C \sum_{j,k,r=1}^{3} \|\partial_{\tau_{jk}} f_r\|_{L^p(\partial\Omega)} \quad \text{if } 1 < p \leq 2, \tag{3.184}$$

$$\|u\|_{F^{p,q}_{2+\frac{1}{p}}(\Omega)} \leq C \sum_{j,k,r=1}^{3} \|\partial_{\tau_{jk}} f_r\|_{L^p(\partial\Omega)} \quad \text{if } 2 \leq p < 2 + \varepsilon, \ 0 < q < \infty. \tag{3.185}$$

Proof. Granted Theorem 3.19, existence follows as in the proof of Theorem 3.6. This latter result also guarantees uniqueness when p is near 2. When $1 - \varepsilon < p < 2$, we use the fact that

$$h^{1,p}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega) \tag{3.186}$$

and

$$\Delta^2 u = 0 \ \text{ and } \ M(\nabla\nabla u) \in L^p(\partial\Omega) \implies M(\nabla u) \in L^{p^*}(\partial\Omega) \tag{3.187}$$

provided $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$. See [39] for the implication (3.187). We may then invoke the uniqueness part in the L^p-Dirichlet problem (3.18) with p near 2 in order to finish the proof of uniqueness for the current problem. Finally, (3.184)-(3.185) can be justified analogously to how we have established (3.70)-(3.71). □

We conclude this subsection with a few comments. First, much as noted in Remark 1 at the end of § 3.1, it is possible to patch together results such as those in Theorems 3.20-3.22, in order to prove similar well-posedness results in *arbitrary* bounded Lipschitz domains in $\mathbb{R}^3$.

Second, similar to what was observed in Remark 2 at the end of § 3.1, both Theorem 3.20 and Theorem 3.22 have suitable counterparts in the *complements* of bounded Lipschitz domains. In each case, the decay condition (2.108) is to be imposed.

3.7 The Dirichlet problem with Hölder and bmo data

Here we discuss the case of the problem (3.18) with data from Hölder and BMO spaces. A few preliminaries are necessary. Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, define the set of *Carleson measures*, *Car* (Ω), as the subclass of Borelian measures μ on Ω satisfying

$$\|\mu\|_{Car\,(\Omega)} := \sup\left\{\frac{\mu(B(X,R)\cap\Omega)}{R^{n-1}} :\ X \in \partial\Omega,\ 0 < R < \mathrm{diam}\,(\partial\Omega)\right\} < \infty. \tag{3.188}$$

Two well-known properties of Carleson measures are going to be of importance for us here. First,

$$f \in L^n(\Omega) \Longrightarrow \mu := |f|\,dX \in Car\,(\Omega) \quad \text{and} \quad \|\mu\|_{Car\,(\Omega)} \leq C\|f\|_{L^n(\Omega)}. \tag{3.189}$$

Second, let $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be an odd function which is homogeneous of degree $-(n-1)$. Also, fix some $b \in L^\infty(\partial\Omega)$ and assume that the operator

$$\mathscr{T} f(X) := \int_{\partial\Omega} k(X-Y)b(Y)f(Y)\,d\sigma(Y), \quad X \in \Omega, \tag{3.190}$$

satisfies

$$\mathscr{T} 1 \equiv const \quad \text{in} \quad \Omega. \tag{3.191}$$

Then, with $\delta(X)$ denoting the distance from X to $\partial\Omega$,

$$\big\|\,(\mathscr{T} f)|_{\partial\Omega}\big\|_{\mathrm{bmo}\,(\partial\Omega)} + \big\|\,|\nabla \mathscr{T} f|^2\delta\,dX\big\|_{Car\,(\Omega)} \leq C\|f\|_{\mathrm{bmo}\,(\partial\Omega)}. \tag{3.192}$$

Next, we introduce a special subclass, $Car_*(\Omega)$, of the space of Carleson measures in Ω, by setting

$$\mu \in Car_*(\Omega) \overset{def}{\Longleftrightarrow} \mu \in Car\,(\Omega) \quad \text{and} \quad \lim_{R_*\to 0}\left(\sup_{\substack{X\in\partial\Omega\\ 0<R<R_*}} \frac{\mu(B(X,R)\cap\Omega)}{R^{n-1}}\right) = 0. \tag{3.193}$$

Then, in the context of (3.190)-(3.191), we also have

$$\big\|\,(\mathscr{T}f)|_{\partial\Omega}\big\|_{\mathrm{vmo}\,(\partial\Omega)}+\big\|\,|\nabla\mathscr{T}f|^2\delta\,dX\big\|_{Car_*(\Omega)}\leq C\|f\|_{\mathrm{vmo}\,(\partial\Omega)}. \tag{3.194}$$

Lemma 3.23. *Assume that $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain and consider the (formal) application*

$$\dot{f}=(f_0,f_1,...,f_n)\mapsto h:=\mathcal{D}f_0-\mathcal{S}\Big(\textstyle\sum_{j=1}^{n}\nu_j f_j\Big). \tag{3.195}$$

Then

$$\big\|\,(\nabla h)|_{\partial\Omega}\big\|_{\mathrm{bmo}\,(\partial\Omega)}+\big\|\,|\nabla\nabla h|^2\delta\,dX\big\|_{Car\,(\Omega)}\leq C\|\dot{f}\|_{WA(\mathrm{bmo}\,(\partial\Omega))}, \tag{3.196}$$

$$\big\|\,(\nabla h)|_{\partial\Omega}\big\|_{\mathrm{vmo}\,(\partial\Omega)}+\big\|\,|\nabla\nabla h|^2\delta\,dX\big\|_{Car_*(\Omega)}\leq C\|\dot{f}\|_{WA(\mathrm{vmo}\,(\partial\Omega))}. \tag{3.197}$$

Also, for each $s\in(0,1)$,

$$\|h\|_{C^{s+1}(\bar{\Omega})}+\sup_{X\in\Omega}\delta(X)^{1-s}|\nabla\nabla h(X)|\leq C\|\dot{f}\|_{WA(C^s(\partial\Omega))}. \tag{3.198}$$

Proof. Let $\dot{f}$ and h be as in the statement of the lemma. Then for each $i\in\{1,...,n\}$, we have

$$\begin{aligned}\partial_i h &= \partial_i\mathcal{D}f_0-\partial_i\mathcal{S}(\nu_j f_j)\\ &= \partial_j\mathcal{S}(\partial_{\tau_{ij}}f_0)-\partial_i\mathcal{S}(\nu_j f_j)\\ &= \partial_j\mathcal{S}(\nu_i f_j-\nu_j f_i)-\partial_i\mathcal{S}(\nu_j f_j)\\ &= \mathcal{D}f_i+\mathscr{R}_{ki}f_k\end{aligned} \tag{3.199}$$

where we have set

$$\mathscr{R}_{ij}f(X):=\int_{\partial\Omega}\partial_{\tau_{ij}(Y)}[\Gamma(X-Y)]f(Y)\,d\sigma(Y),\qquad X\in\Omega. \tag{3.200}$$

Then (3.196) is a consequence of this and (3.190)-(3.192). In fact, (3.197) is handled similarly, based on (3.194). Furthermore, from (3.199) and Proposition 4.21 (stated and proved later on), it follows that

$$\begin{aligned}&\mathcal{D},\,\mathscr{R}_{ij}:C^s(\partial\Omega)\longrightarrow C^s(\bar{\Omega}),\\ &\delta^{1-s}\nabla\mathcal{D},\,\delta^{1-s}\nabla\mathscr{R}_{ij}:C^s(\partial\Omega)\longrightarrow L^\infty(\Omega),\end{aligned} \tag{3.201}$$

are bounded operators for every $s\in(0,1)$. From this, (3.198) follows. □

After this preamble, we are ready to discuss the Dirichlet problem for the bi-Laplacian with Hölder data.

Theorem 3.24. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the property that if $s \in (0, \varepsilon)$ then the problem*

$$\begin{cases} \Delta^2 u = 0 \quad in \quad \Omega, \\ u\Big|_{\partial\Omega} = f_0 \in C^s(\partial\Omega), \\ (\partial_j u)\Big|_{\partial\Omega} = f_j \in C^s(\partial\Omega), \;\; 1 \leq j \leq 3, \end{cases} \tag{3.202}$$

where, necessarily, the family $\dot{f} := (f_0, f_1, \ldots, f_n)$ satisfies the compatibility conditions in (2.21), has a unique solution $u \in C^{s+1}(\bar{\Omega})$. Moreover, there exists a finite constant $\kappa > 0$ such that

$$\|u\|_{C^{s+1}(\bar{\Omega})} + \sup_{X \in \Omega} \delta(X)^{1-s} |\nabla\nabla u(X)| \leq \kappa \sum_{j=0}^{3} \|f_j\|_{C^s(\partial\Omega)}. \tag{3.203}$$

A similar well-posedness result is valid for the exterior version of (3.202), when the decay condition (2.108) is added.

Proof. A solution for (3.202) is constructed using the recipe in the proof of Theorem 3.1, with $g_0 := f_0$ and $g_1 := \nu_1 f_1 + \nu_2 f_2 + \nu_3 f_3$. Lemma 3.23 then ensures that h in (3.8) belongs to $C^{s+1}(\bar{\Omega})$. Since by Lemma 2.13, Proposition 4.21 (with $p = \infty$) and (4.61), the function in (3.9) also belongs to $C^{s+1}(\bar{\Omega})$, it follows that a solution u for (3.202) can be found such that (3.203) holds. Uniqueness is an easy consequence of the corresponding uniqueness part in Theorem 3.1. The estimate for $\|u\|_{C^{s+1}(\bar{\Omega})}$ is implicit in the above reasoning. Then (4.61) and (4.57) allow us to prove (3.203) in full. The exterior version of (3.202), with the decay condition (2.108) included, is dealt with via the Kelvin transform, as discussed in § 2.4. □

We now deal with the Dirichlet problem for Δ^2 with data in BMO.

Theorem 3.25. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$ which is star-like with respect to the origin. Then the problem*

$$\begin{cases} \Delta^2 u = 0 \quad in \quad \Omega, \\ u\Big|_{\partial\Omega} = f_0 \in \mathrm{bmo}\,(\partial\Omega), \\ (\partial_j u)\Big|_{\partial\Omega} = f_j \in \mathrm{bmo}\,(\partial\Omega), \;\; 1 \leq j \leq 3, \end{cases} \tag{3.204}$$

where, necessarily, the family $\dot{f} := (f_0, f_1, \ldots, f_n)$ satisfies the compatibility conditions in (2.21), has a unique solution u such that $|\nabla\nabla u|^2 \delta \, dX \in Car\,(\Omega)$. Furthermore, there exists a finite constant $C > 0$ such that

$$\big\| \, |\nabla\nabla u|^2 \delta \, dX \big\|_{Car\,(\Omega)} \leq C \sum_{j=0}^{3} \|f_j\|_{\mathrm{bmo}\,(\partial\Omega)}. \tag{3.205}$$

The solution u of (3.204) has an integral representation formula as in (3.3)-(3.4) (with $g_0 := f_0$ and $g_1 := \nu_j f_j$) and satisfies

$$|\nabla\nabla u|^2 \delta \, dX \in Car_*(\Omega) \Longleftrightarrow f_j \in \mathrm{vmo}\,(\partial\Omega), \; 0 \leq j \leq 3. \tag{3.206}$$

Finally, a similar well-posedness result is valid for the exterior version of (3.204) (in which case the decay condition (2.108) is included).

Proof. Granted Lemma 3.23 and (3.173)-(3.174), the same type of reasoning as in the proof of Theorem 3.24 applies (with (3.189) used to handle the lower order terms in (2.190)). This proves that (3.204) is well-posed. The right-to-left implication in (3.206) is a consequence of (3.174) and the integral representation formula for u. The left-to-right implication in (3.206) using similar arguments to those in [15]. The exterior version of (3.204) can be dealt with via the Kelvin transform. □

The portion of Theorem 3.25 dealing with BMO data has first been proved in [44], using a different approach (which emphasizes estimates for the associated Green function). In closing, we once again wish to point out that, as far as the well-posedness of the problems in Theorem 3.24 and Theorem 3.25 is concerned, the condition that the domain is star-shaped can, much as before, be eliminated a posteriori.

4 Smoothness spaces

In preparation for discussing the inhomogeneous problem for the bi-Laplacian, in this section we collect a number of results whose general aim is to clarify how the quality of being biharmonic affects the membership to various smoothness spaces.

4.1 Some weighted norm inequalities

For an open set $\Omega \subset \mathbb{R}^n$ set $\delta(X) := \mathrm{dist}\,(X, \partial\Omega)$. In order to facilitate the subsequent discussion, we make the following

Definition 4.1. *Assume that Ω is an open subset of $\mathbb{R}^n$ and set $\delta(X) := dist(X, \partial\Omega)$, $X \in \mathbb{R}^n$. Also, fix $0 < p < \infty$. A function $u \in L^p_{\mathrm{loc}}(\Omega)$ is said to be p-subaveraging if there exists a positive constant C with the following property:*

$$|u(X)| \leq C \left(⨍_{B_r(X)} |u(Y)|^p \, dY \right)^{\frac{1}{p}} \tag{4.1}$$

for almost every $X \in \Omega$ and all $r \in (0, \delta(X))$.

Employing ideas first developed by Fefferman and Stein in [17], the following result can be proved.

Lemma 4.1. *If there exists $p_0 > 0$ such that u is p_0-subaveraging function, then u is p-subaveraging for every $p \in (0, \infty)$.*

Granted this, it is unequivocal to refer to a function u as simply being *sub-averaging* if it is p-sub-averaging for some $p \in (0, \infty)$. The optimal constants which can be used in (4.1) make up what we call the subaveraging character of the function u.

There are clear connections between the subaveraging property and reverse Hölder estimates. To illustrate this, we state the following.

Lemma 4.2. *Let u be a subaveraging function in a domain $\Omega \subset \mathbb{R}^n$ and assume that $0 < p, q < \infty$. Then*

$$\left(\fint_{B_r(X)} |u(Y)|^q \, dY \right)^{\frac{1}{q}} \leq C \left(\fint_{B_{2r}(X)} |u(Y)|^p \, dY \right)^{\frac{1}{p}} \tag{4.2}$$

uniformly for $x \in \Omega$ and $0 < r < \delta(X)/2$, where the constant C depends only on p, q, n and the subaveraging character of u.

Assume next that

$$L = \sum_{|\gamma|=m} a_\gamma \partial^\gamma \tag{4.3}$$

is a (homogeneous) constant-coefficient, differential operator of order $m \in 2\,\mathbb{N}$ in $\mathbb{R}^n$, which is elliptic, in the sense that there exists a finite constant $\Lambda > 1$ such that if $\sigma(L;\xi) := (-1)^{m/2} \sum_{|\gamma|=m} a_\gamma \xi^\gamma$, then

$$\Lambda^{-1}|\xi|^m \leq (-1)^{m/2}\sigma(L;\xi) \leq \Lambda|\xi|^m, \qquad \forall\, \xi \in \mathbb{R}^n. \tag{4.4}$$

or a given open set $\Omega \subset \mathbb{R}^n$, denote by $\operatorname{Ker} L$ the space of all C^∞ functions satisfying $Lu = 0$ in Ω. We present some useful interior estimates, which are essentially folklore.

Lemma 4.3. *Let L be an elliptic differential operator as above and assume that $\Omega \subset \mathbb{R}^n$ is open. Then for each $u \in \operatorname{Ker} L$, $0 < p < \infty$, $k \in \mathbb{N}_o$, and $x \in \Omega$, $0 < r < \delta(X)$,*

$$|\nabla^k u(X)|^p \leq \frac{C}{r^{kp}} \fint_{B_r(X)} |u(Y)|^p \, dY \tag{4.5}$$

where $C = C(L, p, k, n) > 0$ is a finite constant. In particular,

$$u \in \operatorname{Ker} L \Longrightarrow u \text{ is subaveraging.} \tag{4.6}$$

If Ω is a star-like domain with respect to the origin and u is a function defined in Ω, set

$$u_{rad}(X) := \sup_{t>0} |u(e^{-t}X)|, \qquad X \in \bar{\Omega}. \tag{4.7}$$

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, which is star-like domain with respect to the origin, and consider $0 < p < 1$ along with $s > -1/p$. Then for a subaveraging function u in Ω,*

$$\left(\int_\Omega |u_{rad}(X)|^p \, \delta(X)^{sp} \, dX\right)^{1/p} \leq \kappa \left(\int_\Omega |u(X)|^p \, \delta(X)^{sp} \, dX\right)^{1/q} \tag{4.8}$$

where κ depends only on p, s, n, the Lipschitz character of Ω and the subaveraging character of u.

Lemma 4.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that L is an elliptic operator as in (4.3). Then there exists a finite constant $C = C(L, \Omega, p, s, k) > 0$ such that*

$$\Big(\int_\Omega (\delta(X)^{s+k} |\nabla^k u(X)|)^p \, dX\Big)^{1/p} \leq C \Big(\int_\Omega (\delta(X)^s |u(X)|)^p \, dX\Big)^{1/p} \tag{4.9}$$

holds for any $u \in \operatorname{Ker} L$ provided $0 < p < \infty$, $s \in \mathbb{R}$, and $k \in \mathbb{N}_o$.

Lemma 4.6. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and fix k, a nonnegative integer, $0 < q \leq p < \infty$, and $s \in \mathbb{R}$ with $sp > -1$. Also, let L be an elliptic, homogeneous, constant coefficient operator. Then there exists a relatively compact subset $\mathcal{O}$ of Ω such that*

$$\Big(\int_\Omega (\delta(X)^s |u(X)|)^p \, dX\Big)^{1/p} \leq C \left[\Big(\int_\Omega (\delta(X)^{s+k} |\nabla^k u(X)|)^p \, dX\Big)^{1/p} + \sup_{X \in \mathcal{O}} |u(X)|\right], \tag{4.10}$$

uniformly for $u \in \operatorname{Ker} L$.

Let us also record here the following version of Hardy's inequality.

Lemma 4.7. *Assume that $f : \mathbb{R} \to [0, \infty]$ is measurable, $r > 0$ and $0 \leq M \leq \infty$. Then the estimate*

$$\left(\int_0^M \left(\int_x^M y^q f(y) \, dy\right)^p x^{r-1} \, dx\right)^{1/p} \leq C(p, q, r) \left(\int_0^M (y^{q+1} f(y))^p y^{r-1} \, dy\right)^{1/p} \tag{4.11}$$

holds provided either $1 \leq p < \infty$, or $0 < p < 1$ and f is nonincreasing.

Our last lemma here is a result to the effect that, for biharmonic functions, the radial derivative controls the entire gradient in a weighted L^p-sense.

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, star-like with respect to the origin. Also fix $0 < p < \infty$, $a > -1$. Then there exists $C > 0$ such that, for every harmonic function u in Ω,*

$$\int_\Omega \delta(X)^a |\nabla u(X)|^p \, dX \leq C \int_\Omega \delta(X)^a |\nabla_\eta u(X)|^p \, dX. \tag{4.12}$$

Proof. We shall show that there exist a relatively compact neighborhood $\widetilde{\mathcal{O}}$ of the origin in Ω and $C > 0$ such that, for every harmonic function u in Ω,

$$\int_\Omega \delta(x)^a |\nabla u(X)|^p \, dX \leq C \int_\Omega \delta(X)^a |\nabla_\eta u(X)|^p \, dX + C \int_{\widetilde{\mathcal{O}}} |u|^p \, dX. \tag{4.13}$$

Indeed, we claim that (4.13) implies (4.12). To see this, note that the inequality (4.12) is invariant if we renormalize the function u so that $u(0) = 0$. Assume that this is the case and, for $X \in \widetilde{\mathcal{O}}$, write (cf. (2.33)):

$$u(X) = \int_0^1 (\nabla_\eta u)(tX) \, \frac{dt}{t} = \int_0^\varepsilon (\nabla_\eta u)(tX) \, \frac{dt}{t} + \int_\varepsilon^1 (\nabla_\eta u)(tX) \, \frac{dt}{t}, \tag{4.14}$$

where $\varepsilon > 0$ is a small parameter, to be specified momentarily. Hence,

$$|u(X)| \leq C \int_0^\varepsilon |(\nabla u)(tX)| \, dt + C\varepsilon^{-1} \int_0^1 |(\nabla_\eta u)(tX)| \, dt =: A + B. \tag{4.15}$$

Using interior estimates, it follows that for each $X \in \widetilde{\mathcal{O}}$

$$A \leq C\varepsilon \Big(\int_{\mathcal{O}} |\nabla u|^p \Big)^{1/p} \quad \text{and} \quad B \leq C\varepsilon^{-1} \Big(\int_{\mathcal{O}} |\nabla_\eta u|^p \Big)^{1/p}, \tag{4.16}$$

where $\mathcal{O}$ is a slightly larger set than $\widetilde{\mathcal{O}}$ (which can be taken to be independent of ε). Then the majorant of A can be absorbed into the left-hand side of (4.12), granted that $\varepsilon > 0$ is small enough, while the majorant of B is of the right order. This finishes the justification of the fact that it suffices to prove (4.13).

To establish (4.13), we shall first treat the case when $0 < p < 1$. Pick $\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$ two relatively compact open neighborhoods of the origin in Ω, and fix some sufficiently large $M > 0$. Then

$$\int_\Omega \delta(X)^a |\nabla u(X)|^p \, dX = \int_{\Omega \setminus \mathcal{O}} \delta(X)^a |\nabla u(X)|^p \, dX + \int_{\mathcal{O}} \delta(X)^a |\nabla u(X)|^p \, dX \tag{4.17}$$

and note that $\int_{\mathcal{O}} \delta(X)^a |\nabla u(X)|^p \, dX \leq C \int_{\widetilde{\mathcal{O}}} |u|^p \, dX$, by relying on interior estimates. Also, since $a > -1$,

$$\begin{aligned}\int_{\Omega\setminus\mathcal{O}} \delta(X)^a |\nabla u(X)|^p\, dX &= \int_{\Omega\setminus\mathcal{O}} \delta(X)^a \left|\int_0^1 (\nabla_\eta \nabla u)(tX)\, \frac{dt}{t} + \nabla u(0)\right|^p dX \\ &\leq C\int_{\Omega\setminus\mathcal{O}} \delta(X)^a \Big(\int_{e^{-M}}^1 |(\nabla_\eta \nabla u)(tX)|\frac{dt}{t}\Big)^p dX \\ &\quad + C\int_{\Omega\setminus\mathcal{O}} \delta(X)^a \Big(\int_0^{e^{-M}} |(\nabla_\eta \nabla u)(tX)|\frac{dt}{t}\Big)^p dX \\ &\quad + C|\nabla u(0)|^p =: I + II + III. \end{aligned} \tag{4.18}$$

In II, use $|(\nabla_\eta \nabla u)(tX)| \leq C\, t \sup\{|(\nabla\nabla u)(Z)| : Z \in \mathcal{O}\}$ so that, by Lemma 4.3, we have

$$II + III \leq C \int_{\widetilde{\mathcal{O}}} |u|^p\, dX. \tag{4.19}$$

To estimate I, we first observe that interchanging ∇ and ∇_η gives

$$\begin{aligned} I &\leq \int_{\Omega\setminus\mathcal{O}} \delta(X)^a \Big(\int_{e^{-M}}^1 |(\nabla \nabla_\eta u)(tX)|\frac{dt}{t}\Big)^p dX \\ &\quad + \int_{\Omega\setminus\mathcal{O}} \delta(X)^a \Big(\int_{e^{-M}}^1 |(\nabla u)(tX)|\frac{dt}{t}\Big)^p dX =: I' + I''. \end{aligned} \tag{4.20}$$

We now focus our attention on I'. Use polar coordinates to write each $X \in \Omega\setminus\mathcal{O}$ as $X = r\omega$, $\omega \in S^{n-1}$ and $e^{-M}\varphi(\omega) < r < \varphi(\omega)$. Then,

$$I' \leq C \int_{S^{n-1}} \int_{e^{-M}\varphi(\omega)}^{\varphi(\omega)} \delta(r\omega)^a \Big(\int_{e^{-M}}^1 |(\nabla\nabla_\eta u)(tr\omega)|\frac{dt}{t}\Big)^p r^{n-1} dr\, d\omega. \tag{4.21}$$

Next we make the change of variables $r = e^{-s}\varphi(\omega)$, $0 < s < M$, so $\delta(r\omega) = \delta(e^{-s}\varphi(\omega)\omega) \approx (1 - e^{-s}) \approx s$ and $dr \approx ds$, and thus

$$I' \leq C \int_{S^{n-1}} \int_0^M s^a \Big(\int_{e^{-M}}^1 |(\nabla\nabla_\eta u)_{rad}(te^{-s}\varphi(\omega)\omega)|\frac{dt}{t}\Big)^p ds\, d\omega. \tag{4.22}$$

Furthermore, if we let $t = e^{-\lambda}$, $0 < \lambda < M$, we get $\frac{dt}{t} \approx d\lambda$ and (4.22) becomes

$$\begin{aligned} I' &\leq C \int_{S^{n-1}} \int_0^M s^a \Big(\int_0^M |(\nabla\nabla_\eta u)_{rad}(e^{-(\lambda+s)}\varphi(\omega)\omega)|\, d\lambda\Big)^p ds\, d\omega \\ &\leq C \int_{S^{n-1}} \int_0^{2M} s^a \Big(\int_s^{2M} |(\nabla\nabla_\eta u)_{rad}(e^{-\mu}\varphi(\omega)\omega)|\, d\mu\Big)^p ds\, d\omega, \end{aligned} \tag{4.23}$$

where for the last inequality in (4.23) we made the change of variables: $\lambda+s =: \mu$, $t \in [0, M]$. At this point, we use Hardy's inequality from Lemma 4.7 (here $a > -1$ is used) in order to obtain

$$
\begin{aligned}
I' &\leq \int_{S^{n-1}} \int_0^{2M} s^{p+a} |(\nabla\nabla_\eta u)_{rad}(e^{-s}\varphi(\omega)\omega)|^p \, ds \, d\omega \\
&\leq C \int_{\Omega\setminus\mathcal{O}} \delta(X)^{p+a} |(\nabla\nabla_\eta u)_{rad}(X)|^p \, dX. \qquad (4.24)
\end{aligned}
$$

Since $a > -1$ and $p > 0$, we have $p + a > -1$ so Lemma 4.4 applies in concert with (4.24) to give

$$
\begin{aligned}
I' &\leq \int_\Omega \delta(X)^{p+a} |(\nabla\nabla_\eta u)(X)|^p \, dX \\
&\leq C \int_\Omega \delta(X)^a |(\nabla_\eta u)(X)|^p \, dX. \qquad (4.25)
\end{aligned}
$$

We note that for the last inequality in (4.25) we have employed Lemma 4.5 since $\nabla_\eta u$ is harmonic.

In a similar fashion,

$$
I'' \leq C \int_\Omega \delta(X)^a |u(X)|^p dX. \qquad (4.26)
$$

Because u is harmonic, we can repeat the same type of argument to obtain

$$
\int_\Omega \delta(X)^a |u(X)|^p \, dX \leq C \int_\Omega \delta(X)^{a+p} |(\nabla_\eta u)(X)|^p \, dX + \int_{\widetilde{\mathcal{O}}} |u|^p \, dX. \qquad (4.27)
$$

Since $\delta(X)^{a+p} \leq C\delta(X)^a$, the right-hand side above can be dominated by a term of the right order (as far as (4.13) is concerned).

All in all, this proves the lemma in the case when $0 < p < 1$. When $1 \leq p < \infty$, the proof is simpler, in that the involvement of the radial maximal function is unnecessary. □

4.2 Besov and Triebel-Lizorkin spaces in Lipschitz domains

By $B^{p,q}_s(\mathbb{R}^n)$, $F^{p,q}_s(\mathbb{R}^n)$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$, we denote the classical scales of Besov and Triebel-Lizorkin spaces in the Euclidean space. See, e.g., [54], [48].

Next, given an arbitrary open subset Ω of $\mathbb{R}^n$, we denote by $f|_\Omega$ the restriction of a distribution f in $\mathbb{R}^n$ to Ω. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, both $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$ are spaces of (tempered) distributions, hence it is meaningful to define

$$\begin{aligned} &A^{p,q}_s(\Omega) := \{f \text{ distribution in } \Omega :\ \exists\, g \in A^{p,q}_s(\mathbb{R}^n) \ \text{ such that } \ g|_\Omega = f\}, \\ &\|f\|_{A^{p,q}_s(\Omega)} := \inf\{\|g\|_{A^{p,q}_s(\mathbb{R}^n)} :\ g \in A^{p,q}_s(\mathbb{R}^n),\ g|_\Omega = f\}, \qquad f \in A^{p,q}_s(\Omega), \end{aligned} \tag{4.28}$$

where $A = B$, or $A = F$. The existence of an universal extension operator for Besov and Triebel-Lizorkin spaces in an arbitrary Lipschitz domain $\Omega \subset \mathbb{R}^n$ has been solved established by V. Rychkov in [49]. This allows transferring a number of properties of the Besov-Triebel-Lizorkin spaces in the Euclidean space $\mathbb{R}^n$ to the setting of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Here, we only wish to mention a few of these properties. First,

$$B^{p,\min(p,q)}_s(\Omega) \hookrightarrow F^{p,q}_s(\Omega) \hookrightarrow B^{p,\max(p,q)}_s(\Omega) \tag{4.29}$$

whenever $0 < p \leq \infty$, $0 < q < \infty$, and $s \in \mathbb{R}$. Second, if k is a nonnegative integer and $1 < p < \infty$, then

$$F^{p,2}_k(\Omega) = W^{k,p}(\Omega) := \{f \in L^p(\partial\Omega) :\ \partial^\alpha f \in L^p(\partial\Omega),\ |\alpha| \leq k\}, \tag{4.30}$$

Third, if $k \in \mathbb{N}_0$ and $0 < s < 1$, then

$$B^{\infty,\infty}_{k+s}(\Omega) = C^{k+s}(\Omega), \tag{4.31}$$

where

$$\begin{aligned} C^{k+s}(\Omega) \ &:= \ \Big\{u \in C^k(\Omega) :\ \text{with} \ \ \|u\|_{C^{k+s}(\Omega)} < \infty, \ \ \text{where} \\ &\|u\|_{C^{k+s}(\Omega)} := \sum_{j=1}^{k} \|\nabla^j u\|_{L^\infty(\Omega)} + \sum_{|\alpha|=k} \sup_{X \neq Y \in \Omega} \frac{|\partial^\alpha u(X) - \partial^\alpha u(Y)|}{|X-Y|^s}\Big\}. \end{aligned} \tag{4.32}$$

Going further, for $0 < p, q \leq \infty$, $s \in \mathbb{R}$, we set

$$\begin{aligned} &A^{p,q}_{s,0}(\Omega) := \{f \in A^{p,q}_s(\mathbb{R}^n) :\ \operatorname{supp} f \subseteq \overline{\Omega}\}, \\ &\|f\|_{A^{p,q}_{s,0}(\Omega)} := \|f\|_{A^{p,q}_s(\mathbb{R}^n)}, \quad f \in A^{p,q}_{s,0}(\Omega), \end{aligned} \tag{4.33}$$

where, as usual, either $A = F$ and $p < \infty$ or $A = B$. Thus, $B^{p,q}_{s,0}(\Omega)$, $F^{p,q}_{s,0}(\Omega)$ are closed subspaces of $B^{p,q}_{s,0}(\mathbb{R}^n)$ and $F^{p,q}_{s,0}(\mathbb{R}^n)$, respectively.

Finally, for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, we introduce

$$\begin{aligned} &A^{p,q}_{s,z}(\Omega) := \{f \text{ distribution in } \Omega :\ \exists\, g \in A^{p,q}_{s,0}(\Omega) \ \text{ with } \ g|_\Omega = f\}, \\ &\|f\|_{A^{p,q}_{s,z}(\Omega)} := \inf\{\|g\|_{A^{p,q}_s(\mathbb{R}^n)} :\ g \in A^{p,q}_{s,0}(\Omega),\ g|_\Omega = f\}, \qquad f \in A^{p,q}_{s,z}(\Omega), \end{aligned} \tag{4.34}$$

where, as before, $A = F$ and $p < \infty$ or $A = B$.

If $1 < p,q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$, then

$$\Big(A^{p,q}_{s,z}(\Omega)\Big)^* = A^{p',q'}_{-s}(\Omega) \quad \text{if } \ s > -1 + \tfrac{1}{p}, \tag{4.35}$$

$$\Big(A^{p,q}_{s}(\Omega)\Big)^* = A^{p',q'}_{-s,z}(\Omega) \quad \text{if } \ s < \tfrac{1}{p}. \tag{4.36}$$

Furthermore, for each $s \in \mathbb{R}$ and $1 < p,q < \infty$, the spaces $A^{p,q}_s(\Omega)$ and $A^{p,q}_{s,0}(\Omega)$ are reflexive.

There is yet another type of smoothness space which will play a significant role in this paper. Specifically, for $\Omega \subset \mathbb{R}^n$ Lipschitz domain, we set

$$\overset{\circ}{A}{}^{p,q}_s(\Omega) := \text{ the closure of } C^\infty_0(\Omega) \text{ in } A^{p,q}_s(\Omega), \qquad 0 < p,q \le \infty, \ s \in \mathbb{R}, \tag{4.37}$$

where, as usual, $A = F$ or $A = B$. For every $0 < p,q < \infty$ and $s \in \mathbb{R}$, we then have

$$A^{p,q}_{s,z}(\Omega) \hookrightarrow \overset{\circ}{A}{}^{p,q}_s(\Omega) \hookrightarrow A^{p,q}_s(\Omega), \qquad \text{continuously.} \tag{4.38}$$

Proposition 3.1 in [55] ensures that

$$\overset{\circ}{A}{}^{p,q}_s(\Omega) = A^{p,q}_s(\Omega) = A^{p,q}_{s,z}(\Omega), \qquad A \in \{F,B\}, \tag{4.39}$$

whenever $0 < p,q < \infty$, $\max\Big(1/p-1, n(1/p-1)\Big) < s < 1/p$, and $\min\{p,1\} \le q < \infty$ in the case $A = F$.

The following useful extension by zero result appears in [55].

Proposition 4.9. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $0 < p,q \le \infty$ and $\max\Big(1/p-1, n(1/p-1)\Big) < s$. Then extension by zero defined as*

$$\widetilde{f}(X) := \begin{cases} f(X), & \text{if } X \in \Omega, \\ 0, & \text{if } X \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.40}$$

induces a linear and bounded operator from $B^{p,q}_{s,z}(\Omega)$ to $B^{p,q}_{s,0}(\Omega)$ and, if $p < \infty$, from $F^{p,q}_{s,z}(\Omega)$ to $F^{p,q}_{s,0}(\Omega)$.

Furthermore, if $\max\Big(1/p-1, n(1/p-1)\Big) < s < 1/p$ and $0 < p,q < \infty$, this operator also maps $B^{p,q}_s(\Omega)$ to $B^{p,q}_{s,0}(\Omega)$ and, if $\min\{p,1\} \le q$, $F^{p,q}_s(\Omega)$ to $F^{p,q}_{s,0}(\Omega)$.

To continue, recall (4.37). We recall a useful characterization of the spaces in (4.37), established in [36], building on the earlier work in [19].

Proposition 4.10. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $m \in \mathbb{N}$. Also, suppose that $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $\min\{1,p\} \le q < \infty$. Then*

$$\overset{\circ}{F}{}^{p,q}_{m-1+s+1/p}(\Omega) = \Big\{u \in F^{p,q}_{m-1+s+1/p}(\Omega) : \ \mathrm{Tr}\,(\partial^\alpha u) = 0 \ \ \textit{for} \ \ |\alpha| \le m-1\Big\}. \tag{4.41}$$

Furthermore, a similar result is valid for the scale of Besov spaces. More specifically, if $m \in \mathbb{N}$, $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ *and* $0 < q < \infty$, *then*

$$\overset{\circ}{B}{}^{p,q}_{m-1+s+1/p}(\Omega) = \Big\{ u \in B^{p,q}_{m-1+s+1/p}(\Omega) : \operatorname{Tr}(\partial^\alpha u) = 0 \;\; \textit{for} \;\; |\alpha| \leq m-1 \Big\}. \tag{4.42}$$

Recall (4.34). The result below is proved in [36], and extends earlier work in [19].

Proposition 4.11. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$. *Then*

$$\overset{\circ}{F}{}^{p,q}_{m-1+s+1/p}(\Omega) = F^{p,q}_{m-1+s+1/p,z}(\Omega) \tag{4.43}$$

provided $m \in \mathbb{N}_0$, $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ *and* $\min\{1,p\} \leq q < \infty$.

Furthermore,

$$\overset{\circ}{B}{}^{p,q}_{m-1+s+1/p}(\Omega) = B^{p,q}_{m-1+s+1/p,z}(\Omega) \tag{4.44}$$

whenever $m \in \mathbb{N}_0$, $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ *and* $0 < q < \infty$.

We now record a result to the effect that distributions in Besov and Triebel-Lizorkin spaces supported on Lipschitz surfaces actually vanish if the amount of smoothness exhibited on these scales is not too small.

Proposition 4.12. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$. *Then*

$$u \in B^{p,q}_s(\mathbb{R}^n) \quad \textit{and} \quad \operatorname{supp} u \subseteq \partial\Omega \implies u \equiv 0 \;\; \textit{in} \;\, \mathbb{R}^n \tag{4.45}$$

whenever

$$0 < p, q \leq \infty, \qquad \max\Big(\tfrac{1}{p} - 1\,,\, n\big(\tfrac{1}{p} - 1\big)\Big) < s. \tag{4.46}$$

As a consequence, if s, p, q *are as in (4.46) and* $\mathcal{O}$ *is an open subset of* Ω, *then*

$$w \in B^{p,q}_{s,z}(\Omega) \quad \textit{and} \quad \operatorname{supp} w \subseteq \overline{\mathcal{O}} \implies w\Big|_{\mathcal{O}} \in B^{p,q}_{s,z}(\mathcal{O}), \tag{4.47}$$

plus a natural estimate.

Finally, similar conclusions are valid for the scale of Triebel-Lizorkin spaces $F^{p,q}_s(\mathbb{R}^n)$ *if, in addition,* $p < \infty$.

Proof. When Ω is a smooth domain, a proof of (4.45), as well as its counterpart in Triebel-Lizorkin spaces, can be found on pp. 45-46 of [54]. However, an inspection reveals that the argument continues to hold for Lipschitz domains. See also the discussion in [55].

Consider next w as in the left side of (4.47). Then $w = f|_\Omega$ for some $f \in B_s^{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq \overline{\Omega}$. Since $\operatorname{supp} w \subseteq \overline{\mathcal{O}}$, it follows that $\operatorname{supp} f \subseteq \overline{\mathcal{O}} \cup \partial\Omega$. However, due to (4.45) (and localization), this self-improves to $\operatorname{supp} f \subseteq \overline{\mathcal{O}}$. Thus, since $w|_{\mathcal{O}} = f|_{\mathcal{O}}$, the conclusion in (4.47) follows. □

Our next result models the behavior of volume potentials, associated with homogeneous, constant coefficient elliptic differential operators in $\mathbb{R}^n$. A proof appears in [24].

Proposition 4.13. *If L is a homogeneous, constant coefficient, elliptic operator of order m and $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$, then $T := \phi L^{-1} \psi$, where L^{-1} is regarded as a classical pseudodifferential operator of order $-m$, has the following mapping properties:*

$$T : F_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow F_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{4.48}$$

boundedly whenever $0 < p < \infty$, and

$$T : B_\alpha^{p,q}(\mathbb{R}^n) \longrightarrow B_{\alpha-m}^{p,q}(\mathbb{R}^n) \tag{4.49}$$

boundedly, whenever $0 < p \leq \infty$.

Denote by $L^{p,q}(\Omega)$, $0 < p, q \leq \infty$, the Lorentz scale of spaces in some open subset Ω of $\mathbb{R}^n$; cf. [3].

Lemma 4.14. *If Ω is a Lipschitz domain in $\mathbb{R}^n$ then*

$$\begin{aligned} &L^{r,q}(\Omega) \hookrightarrow B_s^{p,q}(\Omega) \quad \textit{provided} \quad \tfrac{1}{r} = \tfrac{1}{p} - \tfrac{s}{n}, \\ &\textit{whenever} \quad p > r > 1 \quad \textit{and} \quad 0 < q \leq \infty. \end{aligned} \tag{4.50}$$

Proof. Assume that $p > r > 1$, $s \in \mathbb{R}$, $0 < q \leq \infty$, and select r_0, r_1, s_0, s_1 and $\theta \in (0,1)$ such that

$$1 < r_0 < r < r_1 < p, \quad \tfrac{1}{r} = \tfrac{1-\theta}{r_0} + \tfrac{\theta}{r_1} \quad \text{and} \quad s = (1-\theta)s_0 + \theta s_1. \tag{4.51}$$

Then, according to (4.29)-(4.30), for $j \in \{0,1\}$ we have

$$L^{r_j}(\Omega) \hookrightarrow B_{s_j}^{p,q_j}(\Omega) \quad \text{granted that} \quad q_j \geq r_j \quad \text{and} \quad \tfrac{1}{r_j} = \tfrac{1}{p} - \tfrac{s_j}{n}. \tag{4.52}$$

Since by real interpolation $(L^{r_0}(\Omega), L^{r_1}(\Omega))_{\theta,q} = L^{r,q}(\Omega)$, the inclusion in (4.50) is a consequence of (4.51)-(4.52) and the fact that $(B_{s_0}^{p,q_0}(\Omega), B_{s_1}^{p,q_1}(\Omega))_{\theta,q} = B_s^{p,q}(\Omega)$. □

4.3 Solutions of elliptic PDE's on Besov-Triebel-Lizorkin spaces

Throughout this subsection, we let L be a homogeneous, elliptic differential operator of even order with (possibly matrix-valued) constant coefficients. Also, fix a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and denote by $\operatorname{Ker} L$ the space of functions u satisfying $Lu = 0$ in Ω.

Denote by $(\cdot,\cdot)_{\theta,p}$ and $[\cdot,\cdot]_\theta$, respectively, the real and complex method of interpolation. The theorem below appears in [30], [24].

Theorem 4.15. *Consider an elliptic, homogeneous, constant coefficient differential operator L and fix a bounded Lipschitz domain Ω in $\mathbb{R}^n$. Also, assume that $0 < q_0, q_1, q \leq \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < \theta < 1$, and set $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then, if $0 < p < \infty$,*

$$\Big(F^{p,q_0}_{\alpha_0}(\Omega) \cap \operatorname{Ker} L\,,\, F^{p,q_1}_{\alpha_1}(\Omega) \cap \operatorname{Ker} L\Big)_{\theta,q} = B^{p,q}_{\alpha}(\Omega) \cap \operatorname{Ker} L, \tag{4.53}$$

and if $0 < p \leq \infty$,

$$\Big(B^{p,q_0}_{\alpha_0}(\Omega) \cap \operatorname{Ker} L\,,\, B^{p,q_1}_{\alpha_1}(\Omega) \cap \operatorname{Ker} L\Big)_{\theta,q} = B^{p,q}_{\alpha}(\Omega) \cap \operatorname{Ker} L. \tag{4.54}$$

Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then

$$\Big[F^{p_0,q_0}_{\alpha_0}(\Omega) \cap \operatorname{Ker} L\,,\, F^{p_1,q_1}_{\alpha_1}(\Omega) \cap \operatorname{Ker} L\Big]_{\theta} = F^{p,q}_{\alpha}(\Omega) \cap \operatorname{Ker} L. \tag{4.55}$$

Finally, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ and either $p_0 + q_0 < \infty$ or $p_1 + q_1 < \infty$, then

$$\Big[B^{p_0,q_0}_{\alpha_0}(\Omega) \cap \operatorname{Ker} L\,,\, B^{p_1,q_1}_{\alpha_1}(\Omega) \cap \operatorname{Ker} L\Big]_{\theta} = B^{p,q}_{\alpha}(\Omega) \cap \operatorname{Ker} L, \tag{4.56}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Recall that given $j \in \mathbb{N}_0$, ∇^j stands for vector of all mixed-order partial derivatives of order j. Then, for $0 < p \leq \infty$ and $s \in \mathbb{R}$, denote by $\mathbb{H}^p_s(\Omega; L)$ the space of functions $u \in \operatorname{Ker} L$ subject to the size/smoothness condition

$$\|u\|_{\mathbb{H}^p_s(\Omega;L)} := \|\delta^{\langle s\rangle - s}|\nabla^{\langle s\rangle} u|\|_{L^p(\Omega)} + \sum_{j=0}^{\langle s\rangle - 1} \|\nabla^j u\|_{L^p(\Omega)} < \infty. \tag{4.57}$$

Here and elsewhere, given $s \in \mathbb{R}$ we set

$$\langle s\rangle := \begin{cases} s \text{ if } s \in \mathbb{N}_0, \\ [s] + 1 \text{ if } s > 0,\ s \notin \mathbb{N}, \\ 0 \text{ if } s < 0, \end{cases} \tag{4.58}$$

where $[\cdot]$ is the integer-part function. That is, $\langle s\rangle$ is the smallest nonnegative integer greater than or equal to s. When Ω is bounded, an equivalent quasi-norm on $\mathbb{H}^p_\alpha(\Omega; L)$ is given by

$$\|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u|\|_{L^p(\Omega)} + \sup_{x\in\mathcal{O}}|u(x)|, \tag{4.59}$$

where $\mathcal{O}$ denotes some fixed compact subset of Ω.

Theorem 4.16. *Let L be as above and let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then for each $s \in \mathbb{R}$ and $p, q \in (0,\infty)$,*

$$\mathbb{H}^p_s(\Omega; L) = F^{p,q}_s(\Omega) \cap \operatorname{Ker} L. \tag{4.60}$$

Also, corresponding to $p = \infty$,

$$\mathbb{H}^\infty_{k+s}(\Omega; L) = B^{\infty,\infty}_{k+s}(\Omega) \cap \operatorname{Ker} L \tag{4.61}$$

for each $k \in \mathbb{N}_0$ and $s \in (0,1)$. As a corollary,

$$F^{p,q}_s(\Omega) \cap \operatorname{Ker} L = B^{p,p}_s(\Omega) \cap \operatorname{Ker} L \tag{4.62}$$

whenever $s \in \mathbb{R}$ and $p, q \in (0,\infty)$.

For $1 < p, q < \infty$, $s > 0$, this has been proved in [19] when $L = \Delta$ and in [1] when $L = \Delta^2$. The present version has been established in [30], [24], [36].

Proposition 4.17. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain which is star-like with respect to the origin. Also, fix $0 < p, q < \infty$ and $\alpha > -1$. Then there exists $C = C(\Omega, \alpha, p, q) > 0$ such that if u is a harmonic function in Ω for which $u(0) = 0$ and $\nabla_\eta u \in B^{p,q}_\alpha(\Omega)$ then $u \in B^{p,q}_{\alpha+1}(\Omega)$ and*

$$\|u\|_{B^{p,q}_{\alpha+1}(\Omega)} \le C\|\nabla_\eta u\|_{B^{p,q}_\alpha(\Omega)}. \tag{4.63}$$

Furthermore, if u is a harmonic function in Ω for which $u(0) = 0$ and $\nabla_\eta u \in F^{p,q}_\alpha(\Omega)$ then $u \in F^{p,q}_{\alpha+1}(\Omega)$ and

$$\|u\|_{F^{p,q}_{\alpha+1}(\Omega)} \le C\|\nabla_\eta u\|_{F^{p,q}_\alpha(\Omega)}. \tag{4.64}$$

Proof. Note that if $\mathcal{O}$ is a relatively compact neighborhood of 0 in Ω, and if u is a harmonic function in Ω, then by Theorem 4.16 and Lemma 4.8

$$
\begin{aligned}
\|u\|_{B^{p,p}_{\alpha+1}(\Omega)} &\leq C\Big(\int_\Omega \delta^{(\langle\alpha\rangle-\alpha)p}|\nabla^{\langle\alpha\rangle+1}u|^p\,dX\Big)^{1/p} + C\sup_{\mathcal{O}}|u| \\
&\leq C\Big(\int_\Omega \delta^{(\langle\alpha\rangle-\alpha)p}|\nabla_\eta\nabla^{\langle\alpha\rangle}u|^p\,dX\Big)^{1/p} + C\sup_{\mathcal{O}}|u| \\
&\leq C\Big(\int_\Omega \delta^{(\langle\alpha\rangle-\alpha)p}|\nabla^{\langle\alpha\rangle}\nabla_\eta u|^p\,dX\Big)^{1/p} \\
&\qquad + C\Big(\int_\Omega \delta^{(\langle\alpha\rangle-\alpha)p}|\nabla^{\langle\alpha\rangle}u|^p\,dX\Big)^{1/p} + C\sup_{\mathcal{O}}|u| \\
&\leq C\|\nabla_\eta u\|_{B^{p,p}_\alpha(\Omega)} + C\|u\|_{B^{p,p}_\alpha(\Omega)}. \qquad (4.65)
\end{aligned}
$$

If $\alpha = k-1+s$ with k nonnegative integer and $0 < s \leq 1$, then iterating the above scheme then yields

$$
\|u\|_{B^{p,p}_{\alpha+1}(\Omega)} \leq C\|\nabla_\eta u\|_{B^{p,p}_\alpha(\Omega)} + C\|u\|_{B^{p,p}_{s-1}(\Omega)}. \qquad (4.66)
$$

On the other hand, since u is harmonic, (4.60) with $p = q$ gives

$$
\|u\|_{B^{p,p}_{s-1}(\Omega)} \leq C\|\delta^{1-s}u\|_{L^p(\Omega)} + C\sup_{\mathcal{O}}|u|. \qquad (4.67)
$$

Then Lemma 4.6, Lemma 4.8 and interior estimates allow us to write

$$
\begin{aligned}
\|\delta^{1-s}u\|_{L^p(\Omega)} &\leq C\|\delta^{2-s}|\nabla u|\|_{L^p(\Omega)} + C\sup_{\mathcal{O}}|u| \\
&\leq C\|\delta^{2-s}|\nabla_\eta u|\|_{L^p(\Omega)} + C\sup_{\mathcal{O}}|u| \\
&\leq C\|\delta^{2-s+\langle\alpha\rangle}|\nabla^{\langle\alpha\rangle}\nabla_\eta u|\|_{L^p(\Omega)} + C\sup_{\mathcal{O}}|u| \\
&\leq C\|\delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}\nabla_\eta u|\|_{L^p(\Omega)} + C\sup_{\mathcal{O}}|u| \\
&\leq C\|\nabla_\eta u\|_{B^{p,p}_\alpha(\Omega)} + C\sup_{\mathcal{O}}|u|, \qquad (4.68)
\end{aligned}
$$

since, given that Ω is bounded, $\delta^{2-s+\langle\alpha\rangle} \leq C\delta^{\langle\alpha\rangle-\alpha}$. All together, the above estimates give that

$$
\|u\|_{B^{p,p}_{\alpha+1}(\Omega)} \leq C\|\nabla_\eta u\|_{B^{p,p}_\alpha(\Omega)} + C\sup_{\mathcal{O}}|u|. \qquad (4.69)
$$

At this stage, proceeding as in (4.14)-(4.16) and also making use of interior estimates, for each small $\varepsilon > 0$ it is possible to estimate $\sup_{\mathcal{O}}|u| \leq \varepsilon\|u\|_{L^p(\Omega)} + C_\varepsilon\|\nabla_\eta u\|_{L^p(\Omega)}$, uniformly for harmonic functions u satisfying $u(0) = 0$. Given that $\alpha > -1$, the term $\varepsilon \|u\|_{L^p(\Omega)}$ can then be absorbed into $\|u\|_{B^{p,p}_{\alpha+1}(\Omega)}$, whereas $\|\nabla_\eta u\|_{L^p(\Omega)}$ combines well with $\|\nabla_\eta u\|_{B^{p,p}_{\alpha}(\Omega)}$. The resulting estimate is then precisely (4.63) with $p = q$.

In turn, (4.63) with $p = q$ shows that if $\alpha > -1$ and $0 < p < \infty$ then

$$\nabla_\eta : \{u \in B^{p,p}_{\alpha+1}(\Omega) \cap \mathrm{Ker}\, \Delta : \, u(0) = 0\} \longrightarrow \{u \in B^{p,p}_{\alpha}(\Omega) \cap \mathrm{Ker}\, \Delta : \, u(0) = 0\} \quad (4.70)$$

isomorphically. Interpolating by the real method (cf. Theorem 4.15) then yields that

$$\nabla_\eta : \{u \in B^{p,q}_{\alpha+1}(\Omega) \cap \mathrm{Ker}\, \Delta : \, u(0) = 0\} \longrightarrow \{u \in B^{p,q}_{\alpha}(\Omega) \cap \mathrm{Ker}\, \Delta : \, u(0) = 0\} \quad (4.71)$$

isomorphically, whenever $\alpha > -1$ and $0 < p, q < \infty$. From this, (4.63) follows in full generality. Finally, (4.64) is a consequence of (4.63) and Theorem 4.16.

4.4 Traces and extensions

Here we discuss a number of trace results relevant for the present work. First, we record the following theorem from [30], which extends work done in [19].

Theorem 4.18. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p, s satisfy $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. Then the following hold:*

(i) The restriction to the boundary extends to a linear, bounded operator

$$\mathrm{Tr} : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_{s}(\partial\Omega) \quad \textit{for} \quad 0 < q \leq \infty. \quad (4.72)$$

Moreover, for this range of indices, Tr *is onto and has a bounded right inverse*

$$\mathrm{Ex} : B^{p,q}_{s}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega). \quad (4.73)$$

(ii) Similar considerations hold for

$$\mathrm{Tr} : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s}(\partial\Omega) \quad (4.74)$$

with the convention that $q = \infty$ if $p = \infty$. More specifically, Tr *in (4.74) is linear, bounded, operator which has a linear, bounded right inverse*

$$\mathrm{Ex} : B^{p,p}_{s}(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega). \quad (4.75)$$

To state our next result, recall (2.71)-(2.72).

Theorem 4.19. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p, s satisfy $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Then the following hold:*

(i) The mapping

$$C^1(\bar{\Omega}) \ni u \mapsto \big(u|_{\partial\Omega}, (\partial_1 u)|_{\partial\Omega}, ..., (\partial_n u)|_{\partial\Omega}\big) \in C^0(\partial\Omega) \oplus \cdots \oplus C^0(\partial\Omega) \tag{4.76}$$

extends to a bounded operator

$$\mathrm{Tr}_{\#} : B^{p,q}_{s+\frac{1}{p}+1}(\Omega) \longrightarrow \mathit{WA}\big(B^{p,q}_s(\partial\Omega)\big) \quad \textit{for} \quad 0 < q \leq \infty. \tag{4.77}$$

Moreover, for this range of indices, Tr_* *is onto and has a bounded right inverse*

$$\mathrm{Ex}_{\#} : \mathit{WA}\big(B^{p,q}_s(\partial\Omega)\big) \longrightarrow B^{p,q}_{s+\frac{1}{p}+1}(\Omega). \tag{4.78}$$

(ii) Similar considerations hold for

$$\mathrm{Tr}_{\#} : F^{p,q}_{s+\frac{1}{p}+1}(\Omega) \longrightarrow \mathit{WA}\big(B^{p,p}_s(\partial\Omega)\big) \tag{4.79}$$

with the convention that $q = \infty$ if $p = \infty$. More specifically, Tr *in (4.74) is linear, bounded, operator which has a linear, bounded right inverse*

$$\mathrm{Ex}_{\#} : \mathit{WA}\big(B^{p,p}_s(\partial\Omega)\big) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{4.80}$$

The case when $1 \leq p = q \leq \infty$ for the Besov scale and $1 < p < \infty$, $q = 2$ for the Triebel-Lizorkin scale has been dealt with in [1]. In the current formulation, this is a particular case of a more general trace result appearing in [36]. The same comments apply to our next theorem, stated below.

Theorem 4.20. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $s_0, s_1 \in \mathbb{R}$ and $0 < p, q, q_0, q_1 \leq \infty$ are such that $(n-1)\big(\frac{1}{p}-1\big)_+ < s_0 \neq s_1 < 1$ Then, with $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$,*

$$\Big(\mathit{WA}\big(B^{p,q_0}_{s_0}(\partial\Omega)\big)\,,\, \mathit{WA}\big(B^{p,q_1}_{s_1}(\partial\Omega)\big)\Big)_{\theta,q} = \mathit{WA}\big(B^{p,q}_s(\partial\Omega)\big). \tag{4.81}$$

Furthermore, if $0 < p_i, q_i \leq \infty$, $i = 0, 1$, with $\min\{q_0, q_1\} < \infty$ and $s_0, s_1 \in \mathbb{R}$ are such that $(n-1)\big(\frac{1}{p_i}-1\big)_+ < s_i < 1$, $i = 0, 1$, then

$$\Big[\mathit{WA}\big(B^{p_0,q_0}_{s_0}(\partial\Omega)\big)\,,\, \mathit{WA}\big(B^{p_1,q_1}_{s_1}(\partial\Omega)\big)\Big]_{\theta} = \mathit{WA}\big(B^{p,q}_s(\partial\Omega)\big), \tag{4.82}$$

where $0 < \theta < 1$, $s := (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Finally, if $1 < p < \infty$, $0 < q \le \infty$, $\theta \in (0,1)$, $s_0, s_1 \in \mathbb{R}$, $s_0 \ne s_1$ *and we set* $s = (1-\theta)s_0 + \theta s_1$, *then*

$$\Big(WA\big(F^{p,q_0}_{s_0}(\partial\Omega)\big)\,,\,WA\big(F^{p,q_1}_{s_1}(\partial\Omega)\big)\Big)_{\theta,q} = WA\big(B^{p,q}_{s}(\partial\Omega)\big) \tag{4.83}$$

if either $2 \le q_0, q_1 \le \infty$ *and* $0 \le s_0, s_1 \le 1$, *or* $\min\{p,2\} \le q_0, q_1 \le \infty$ *and* $0 < s_0, s_1 < 1$.

4.5 Singular integral operators on Besov-Triebel-Lizorkin spaces

In this subsection we discuss results describing mapping properties on Besov and Triebel-Lizorkin spaces of integral operators modeled upon the harmonic layer potentials. The first two propositions are proved in [30], and mimic the behavior of the double and single harmonic layer, respectively.

Proposition 4.21. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^n$ *and consider the integral operator*

$$Tf(X) = \int_{\partial\Omega} k(X,Y)f(Y)\,d\sigma(Y), \qquad X \in \Omega, \tag{4.84}$$

satisfying the following conditions:

$$(1)\quad T1 = const, \tag{4.85}$$

$$(2)\quad |\nabla^j_X k(X,Y)| \le C|X-Y|^{-(n+j-1)}, \quad j = 1,2,...,N, \tag{4.86}$$

for some positive integer N. *Then, with* $\delta := \operatorname{dist}(\cdot,\partial\Omega)$,

$$\|\delta^{j-\frac{1}{p}-s}|\nabla^j Tf|\|_{L^p(\Omega)} + \sum_{i=0}^{j-1}\|\nabla^i Tf\|_{L^p(\Omega)} \le C\|f\|_{B^{p,p}_s(\partial\Omega)}, \tag{4.87}$$

granted that $\frac{n-1}{n} < p \le \infty$ *and* $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

Proposition 4.22. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$, *and consider the integral operator*

$$\mathscr{R}f(X) := \int_{\partial\Omega} k(X,Y)f(Y)\,d\sigma(Y), \qquad X \in \Omega, \tag{4.88}$$

whose kernel satisfies the estimates

$$|\nabla^i_X \nabla^j_Y k(X,Y)| \le C|X-Y|^{-(n-2+i+j)}, \quad j = 0,1, \quad 1 \le i \le N, \tag{4.89}$$

for some positive integer N. Also, recall that $\delta(X) := \mathrm{dist}\,(X, \partial\Omega)$. Then

$$\|\delta^{i-\frac{1}{p}-s}|\nabla^i \mathscr{R} f|\|_{L^p(\Omega)} + \sum_{j=0}^{i-1} \|\nabla^j \mathscr{R} f\|_{L^p(\Omega)} \leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega)}, \quad i = 1, 2, ..., N, \tag{4.90}$$

granted that $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$.

Finally, we consider the action of singular integral operators mapping from the Lebesgue scale into Besov spaces.

Proposition 4.23. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and consider a classical pseudodifferential operator $Q(X, D)$ of order $-1 - m$, with $m \in \mathbb{N}$, whose principal symbol, $q(X, \xi)$, is even in ξ if m is odd, and is odd in ξ if m is even. Let k be the Schwartz kernel of $\nabla_X Q(X, D)$, and define*

$$\mathscr{Q} f(X) := \int_{\partial\Omega} k(X - Y, Y) f(Y)\, d\sigma(Y), \qquad X \in \Omega. \tag{4.91}$$

Then, with $a \vee b := \max\{a, b\}$,

$$\mathscr{Q} : L^p(\partial\Omega) \longrightarrow B^{p,p\vee 2}_{m-1+1/p}(\Omega) \tag{4.92}$$

is a bounded operator, for each $p \in (1, \infty)$.

Proof. The result corresponding to $m = 1$ has been proved in [37]. Then matters can be reduced to this case working with $\nabla_X^{m-1} Q(X, D)$ in place of $Q(X, D)$, and using a lifting result to the effect that, for any distribution u in Ω,

$$\partial^\alpha u \in A^{p,q}_s(\Omega), \quad \forall\, \alpha : |\alpha| = k \implies u \in A^{p,q}_{s+k}(\Omega), \tag{4.93}$$

plus natural estimates, granted that $1 < p, q < \infty$, $k \in \mathbb{N}$ and $s \in \mathbb{R}$ (with the convention that $A \in \{B, F\}$). See [35]. □

4.6 Stability and extrapolation on complex interpolation scales

Fix $m \in \mathbb{N}$ and let $U \subset \mathbb{R}^m$ be a convex set. Call a family of analytically convex quasi-Banach spaces $\{X_w\}_{w \in U}$ a *complex interpolation scale* (indexed by U) if for every $\theta \in (0, 1)$ there holds

$$\big[X_{w_0}\,,\, X_{w_1}\big]_\theta = X_{w_\theta}, \quad \text{where } \; w_\theta := (1-\theta) w_0 + \theta w_1. \tag{4.94}$$

In what follows, we shall assume that there is a bound on the moduli of concavity for the spaces X_w, which is uniform in $w \in U$, and that the $\cap_{w \in U} X_w$ is rich; see the discussion in [23]. Important examples of complex interpolation scales for us here are:

$$F^{p,q}_s(\Omega) \text{ and } F^{p,q}_{s,0}(\Omega), \quad \text{indexed by} \;\; s\in\mathbb{R},\;\; 0<p<\infty,\;\; 0<q\leq\infty, \tag{4.95}$$

$$B^{p,q}_s(\Omega) \text{ and } B^{p,q}_{s,0}(\Omega), \quad \text{indexed by} \;\; s\in\mathbb{R},\;\; 0<p<\infty,\;\; 0<q<\infty, \tag{4.96}$$

where Ω is either $\mathbb{R}^n$, or a bounded Lipschitz domain in $\mathbb{R}^n$. Observe that by Proposition 4.12, the restriction operator induces an isomorphism

$$R_\Omega : A^{p,q}_{s,0}(\Omega) \longrightarrow A^{p,q}_{s,z}(\Omega), \tag{4.97}$$

if $A\in\{B,F\}$, provided (4.46) holds, with the convention that $p<\infty$ if $A=F$. From this and (4.95)-(4.96), we may conclude that

$$A^{p,q}_{s,z}(\Omega), \quad \text{for} \;\; 0<p<\infty,\;\; 0<q\leq\infty,\;\; s>\max\Big(\tfrac{1}{p}-1\,,\,n\big(\tfrac{1}{p}-1\big)\Big), \tag{4.98}$$

(with $q<\infty$ if $A=B$) are also complex interpolation scales. Another useful result of this nature is contained in the proposition below.

Proposition 4.24. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain. Then the family of spaces*

$$\Big\{u\in F^{p,q}_{s+\frac{1}{p}+1}(\Omega)\;:\;\mathrm{Tr}_{\#}u=0 \quad on \quad \partial\Omega\Big\}, \tag{4.99}$$

indexed by $0<p,q<\infty$, $(n-1)(\frac{1}{p}-1)_+<s<1$, is a complex interpolation scale.

Furthermore, a similar result is valid in the context of Besov spaces.

As a preliminary matter, we isolate an abstract interpolation result used in the proof of Proposition 4.24.

Lemma 4.25. *Let X_i, Y_i, Z_i, $i=0,1$, be quasi-Banach spaces such that $X_0\cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0, Z_1. Suppose that $Y_i\hookrightarrow Z_i$, $i=0,1$ and there exists a linear operator D such that $D:X_i\to Z_i$ boundedly for $i=0,1$. Define the spaces*

$$X_i(D) := \{u\in X_i :\; Du\in Y_i\}, \quad i=0,1, \tag{4.100}$$

equipped with the graph norm, i.e. $\|u\|_{X_i(D)} := \|u\|_{X_i}+\|Du\|_{Y_i}$, $i=0,1$. Finally, suppose that there exist continuous linear mappings $G:Z_i\to X_i$ and $K:Z_i\to Y_i$ with the property $D\circ G=I+K$ on the spaces Z_i for $i=0,1$. Then, for each $0<\theta<1$ and $0<q\leq\infty$,

$$(X_0(D),X_1(D))_{\theta,q} = \{u\in(X_0,X_1)_{\theta,q} :\; Du\in(Y_0,Y_1)_{\theta,q}\}. \tag{4.101}$$

Furthermore, if the spaces X_0+X_1 and Y_0+Y_1 are analytically convex, then

$$[X_0(D),X_1(D)]_\theta = \{u\in[X_0,X_1]_\theta :\; Du\in[Y_0,Y_1]_\theta\}, \qquad \theta\in(0,1). \tag{4.102}$$

In the context of complex interpolation method for Banach spaces, this appeared in [27]. The present formulation is taken from [24].

Proof of Proposition 4.24. Fix $0 < p_i, q_i < \infty$, $(n-1)(\frac{1}{p_i}-1)_+ < s_i < 1$, $i = 0, 1$, and assume that $\theta \in (0,1)$. Set $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, $s = (1-\theta)s_0 + \theta s_1$. The idea now is to implement Lemma 4.25 for the spaces

$$X_i := F^{p_i,q_i}_{s_i+\frac{1}{p_i}+1}(\Omega), \quad Z_i := WA\big(B^{p_i,p_i}_{s_i}(\partial\Omega)\big), \quad Y_i := 0, \quad i = 0, 1, \tag{4.103}$$

and operators

$$D := \mathrm{Tr}_\#, \qquad G := \mathrm{Ex}_\#, \qquad K := 0. \tag{4.104}$$

Since, by Theorem 4.19, the identity $D \circ G = I + K$ is verified, (4.102) then readily gives the desired conclusion. □

Proposition 4.26. *Assume that $\{X_w\}_{w\in U}$, $\{Y_w\}_{w\in U}$ are two complex interpolation scales, indexed by a common open, convex set $U \subset \mathbb{R}^m$. Also, let T be a linear operator mapping X_w boundedly into Y_w, for each $w \in U$, and denote by T_w this manifestation of T. Set $\mathcal{O} := \{w \in U : T_w$ is invertible$\}$.*

Then $\mathcal{O}$ is open and, if $\mathcal{O}'$ is a convex subset of $\mathcal{O}$, the inverses $T^{-1}_{w_0}$ and $T^{-1}_{w_1}$ agree on $Y_{w_0} \cap Y_{w_1}$ for every $w_0, w_1 \in \mathcal{O}'$.

Proof. The case $m = 1$ is discussed in [23], [24]. When $m \geq 1$, the idea is to make suitable use of the one-dimensional results on line segments $L \subset U$. Since the latter come with estimates that are uniform with respect to the *direction* of L, the desired results follow. □

4.7 Envelopes of non-locally convex spaces

The results in this subsection are from [34], [30]. Let X be a quasi-normed space and, for each $0 < p \leq 1$, let $B_{X,p}$ be the absolutely p-convex hull of the unit ball in X, i.e.,

$$B_{X,p} := \Big\{\sum_{j=1}^n \lambda_j a_j \; : \; a_j \in X,\ \|a_j\|_X \leq 1,\ \sum_{j=1}^n |\lambda_j|^p \leq 1,\ n \in \mathbb{N}\Big\}. \tag{4.105}$$

Set

$$|||x|||_p := \inf\Big\{\lambda > 0 \; : \; x/\lambda \in B_{X,p}\Big\}. \tag{4.106}$$

Then, for each quasi-normed space X whose dual separates its points, we denote by $\mathcal{E}_p(X)$ the *p-envelope* of X, defined as the completion of X in the quasi-norm $|||\cdot|||_p$. The case $p = 1$ corresponds to taking the Banach envelope, i.e. the minimal enlargement of the space in question to a Banach space; cf. [25] for a discussion.

Two results pertaining to this concept are going to be of importance for us here. The first one essentially asserts that for a linear operator, being bounded, and being onto are stable properties under taking envelopes.

Proposition 4.27. *Let X, Y be two quasi-normed spaces and let $T : X \longrightarrow Y$ be a bounded, linear operator. Then, for each $0 < p \leq 1$, this extends to a bounded, linear operator $\widehat{T} : \mathcal{E}_p(X) \longrightarrow \mathcal{E}_p(Y)$. Furthermore,*

$$T : X \longrightarrow Y \text{ onto} \Longrightarrow \widehat{T} : \mathcal{E}_p(X) \longrightarrow \mathcal{E}_p(Y) \text{ onto, and} \tag{4.107}$$

$$T : X \longrightarrow Y \text{ isomorphism} \Longrightarrow \widehat{T} : \mathcal{E}_p(X) \longrightarrow \mathcal{E}_p(Y) \text{ isomorphism.} \tag{4.108}$$

The second result explicitly identifies the envelopes of Besov and Hardy spaces on boundaries of Lipschitz domains.

Theorem 4.28. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\frac{n-1}{n} < p, q \leq p^* \leq 1$ and assume $(n-1)(\frac{1}{p} - 1) < s < 1$. Then*

$$\mathcal{E}_{p^*}(B^{p,q}_{s-1}(\partial\Omega)) = B^{p^*,p^*}_{s^*-1}(\partial\Omega), \quad \text{where} \quad s^* := s + (n-1)(\tfrac{1}{p^*} - \tfrac{1}{p}), \tag{4.109}$$

$$\mathcal{E}_{p^*}(B^{p,q}_{s}(\partial\Omega)) = B^{p^*,p^*}_{s^*}(\partial\Omega), \quad \text{where} \quad s^* := s + (n-1)(\tfrac{1}{p^*} - \tfrac{1}{p}), \tag{4.110}$$

$$\mathcal{E}_{p^*}(h^{p}_{\rm at}(\partial\Omega)) = B^{p^*,p^*}_{s^*}(\partial\Omega), \quad \text{where} \quad s^* := (n-1)(\tfrac{1}{p^*} - \tfrac{1}{p}), \tag{4.111}$$

$$\mathcal{E}_{p^*}(h^{1,p}_{\rm at}(\partial\Omega)) = B^{p^*,p^*}_{s^*}(\partial\Omega), \quad \text{where} \quad s^* := 1 + (n-1)(\tfrac{1}{p^*} - \tfrac{1}{p}). \tag{4.112}$$

5 Main results

5.1 The inhomogeneous problem

For $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$ and $(n-1)\Big(\frac{1}{p} - 1\Big)_+ < s < 1$, set

$$B^{p,q}_{s-1,0}(\partial\Omega) := \{f \in B^{p,q}_{s-1}(\partial\Omega) : \langle f, 1\rangle = 0\}. \tag{5.1}$$

Our first theorem in this subsection is a variant of Theorem 3.19 at the level of Besov spaces.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists $\varepsilon \in (0,1)$ such that the operator $\widehat{\Phi}$ in (3.107) extends isomorphically as*

$$\begin{aligned}\widehat{\Phi} : \Big[B^{p,q}_{s}(\partial\Omega)/\mathbb{R}\Big] \oplus B^{p,q}_{s-1}(\partial\Omega) \oplus B^{p,q}_{s-1,0}(\partial\Omega) \\ \longrightarrow \Big[B^{p,q}_{s}(\partial\Omega)/\langle\, |\eta|\,\rangle\Big] \oplus B^{p,q}_{s-1,0}(\partial\Omega) \oplus B^{p,q}_{s-1}(\partial\Omega)\end{aligned} \tag{5.2}$$

whenever $0 < q \leq \infty$ *and*

$$\tfrac{2}{2+\varepsilon} < p < 1, \quad 1-\varepsilon < s < 1, \quad 2\Big(\tfrac{1}{p}-1\Big) < s-1+\varepsilon. \tag{5.3}$$

Proof. A combination of Theorem 3.19, Proposition 4.27 and Theorem 4.28 gives that the operator in (5.2) is an isomorphism when s, p are as in (5.3) and $q = p$. The extension to the case when $0 < q \leq \infty$ then follows from this and real interpolation. □

Once Theorem 5.1 has been established, we are prepared to deal with the inhomogeneous problem associated with (3.28), when the data is selected from Besov and Triebel-Lizorkin spaces.

Theorem 5.2. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists* $\varepsilon = \varepsilon(\Omega) > 0$ *such that if* $0 < q \leq \infty$ *and* s, p *are as in (5.3) then the problem*

$$\begin{cases} \Delta^2 u = G \in B^{p,q}_{s+\frac{1}{p}-3}(\Omega) \;\; \text{in } \Omega, \\[2mm] \dfrac{\eta_j}{|\eta|}\mathrm{Tr}\,(\partial_j u) = g_0 \in B^{p,q}_{s}(\partial\Omega), \\[2mm] \partial_{\tau_{jk}}\Big(\dfrac{\eta_k}{|\eta|}\mathrm{Tr}\,(\partial_j u)\Big) = g_1 \in B^{p,q}_{s-1,0}(\partial\Omega), \\[2mm] u \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega), \quad \int_\Omega u(X)\,dX = 0, \end{cases} \tag{5.4}$$

has a unique solution. This solution can be represented as

$$u(X) = G_{\#}(X) + H(X) - 2\Gamma(\nabla_\eta \mathcal{S} f)(X) + C_1 + C_2|X|^2, \qquad X \in \Omega, \tag{5.5}$$

where

$$G_{\#} \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega) \text{ is such that } \Delta^2 G_{\#} = G \text{ in } \Omega, \tag{5.6}$$

$C_1, C_2 \in \mathbb{R}$ *are suitably selected constants, and* H *is as in (3.66)-(3.67) for a choice of* $f_1 \in B^{p,q}_s(\partial\Omega)$*,* $f_2 \in B^{p,q}_{s-1}(\partial\Omega)$*, and* $f \in B^{p,q}_{s-1,0}(\partial\Omega)$ *such that*

$$([f_1], f_2, f) = \widehat{\Phi}^{-1}\Big(\Big[g_0 - \tfrac{\eta_j}{|\eta|}\mathrm{Tr}\,(\partial_j G_{\#})\Big]\,,\, g_1 - \partial_{\tau_{jk}}\Big(\tfrac{\eta_k}{|\eta|}\mathrm{Tr}\,(\partial_j G_{\#})\Big)\,,\, 0\Big). \tag{5.7}$$

Furthermore, there exists $C > 0$*, independent* G, g_0, g_1 *such that*

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}+1}(\Omega)} \leq C\Big(\|G\|_{B^{p,q}_{s+\frac{1}{p}-3}(\Omega)} + \|g_0\|_{B^{p,q}_s(\partial\Omega)} + \|g_1\|_{B^{p,q}_{s-1}(\partial\Omega)}\Big). \tag{5.8}$$

Finally, similar results hold for the version of (5.4) with a right-hand side in Triebel-Lizorkin spaces, i.e., for

$$\begin{cases} \Delta^2 u = G \in F^{p,q}_{s+\frac{1}{p}-3}(\Omega) \quad \textit{in } \Omega, \\ \dfrac{\eta_j}{|\eta|}\mathrm{Tr}\,(\partial_j u) = g_0 \in B^{p,p}_{s}(\partial\Omega), \\ \partial_{\tau_{jk}}\Big(\dfrac{\eta_k}{|\eta|}\mathrm{Tr}\,(\partial_j u)\Big) = g_1 \in B^{p,p}_{s-1,0}(\partial\Omega), \\ u \in F^{p,q}_{s+\frac{1}{p}+1}(\Omega), \quad \int_\Omega u(X)\,dX = 0. \end{cases} \tag{5.9}$$

Proof. Existence for (5.4) follows as in the proof of Theorem 3.6, making use of Theorem 5.1, as soon as we show that the function u in (5.5) belongs to $B^{p,q}_{s+\frac{1}{p}+1}(\Omega)$. To show that $H \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega)$ we use the fact that

$$\mathcal{D} : B^{p,q}_{s}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega), \qquad \mathcal{S} : B^{p,q}_{s-1}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega) \tag{5.10}$$

are bounded operators in order to write

$$f_1 \in B^{p,q}_{s}(\partial\Omega), \quad f_2 \in B^{p,q}_{s-1}(\partial\Omega) \Longrightarrow h \in B^{p,q}_{s+\frac{1}{p}}(\Omega) \Longrightarrow \nabla_\eta H \in B^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{5.11}$$

From this and Proposition 4.17, we then infer that $H \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega)$, as desired. Furthermore,

$$f \in B^{p,q}_{s-1}(\partial\Omega) \Longrightarrow \nabla_\eta \mathcal{S} f \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega) \Longrightarrow \Gamma(\nabla_\eta \mathcal{S} f) \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega), \tag{5.12}$$

by Proposition 4.9 and Proposition 4.13 (alternatively, one can use the representation (2.145) and the mapping properties of the operators in the right-hand side of that identity). Combining these facts we may eventually conclude that $u \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega)$.

To prove uniqueness, it suffices to observe that if s, p are as in (5.3) then

$$B^{p,q}_{s}(\partial\Omega) \hookrightarrow B^{1,1}_{\alpha}(\partial\Omega), \quad B^{p,q}_{s-1}(\partial\Omega) \hookrightarrow B^{1,1}_{\alpha-1}(\partial\Omega), \quad B^{p,q}_{s+\frac{1}{p}+1}(\partial\Omega) \hookrightarrow B^{1,1}_{\alpha+2}(\partial\Omega) \tag{5.13}$$

for some $\alpha \in (0,1)$ near 1 so, consequently, the uniqueness result from [1] applies. Finally, the arguments for the problem (5.9) are similar. □

Next, we discuss the inhomogeneous version of (3.18) on Besov-Triebel-Lizorkin spaces in three-dimensional star-like Lipschitz domains. This can be thought of as the local version on Theorem 1.1.

Theorem 5.3. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the following property. Assume that $0 < q \leq \infty$ and that s, p are such that either of the following two conditions*

$$\begin{array}{ll} (I): & 0 \leq \frac{1}{p} < \frac{s}{2} + \frac{1+\varepsilon}{2} \quad \text{and} \quad 0 < s < \varepsilon, \\ (II): & -\frac{\varepsilon}{2} < \frac{1}{p} - \frac{s}{2} < \frac{1+\varepsilon}{2} \quad \text{and} \quad \varepsilon \leq s < 1, \end{array} \tag{5.14}$$

holds. Then the problem

$$\begin{cases} \Delta^2 u = G \in B^{p,q}_{s+\frac{1}{p}-3}(\Omega), \\ \mathrm{Tr}\, u = f_0 \in B^{p,q}_s(\partial\Omega), \\ \mathrm{Tr}\,(\partial_j u) = f_j \in B^{p,q}_s(\partial\Omega), \ \ 1 \leq j \leq 3, \end{cases} \tag{5.15}$$

where, necessarily,

$$\dot{f} := (f_0, f_1, f_2, f_3) \text{ satisfies the compatibility conditions in (2.21)}, \tag{5.16}$$

has a unique solution $u \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega)$. This satisfies

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}+1}(\Omega)} \leq C\Big(\|G\|_{B^{p,q}_{s+\frac{1}{p}-3}(\Omega)} + \sum_{j=0}^{3} \|f_j\|_{B^{p,q}_s(\partial\Omega)}\Big), \tag{5.17}$$

for some finite constant $C = C(\Omega, s, p) > 0$.

Furthermore, similar results are valid for the version of the above boundary problem phrased on Triebel-Lizorkin spaces (with $p, q < \infty$), that is, for

$$\begin{cases} \Delta^2 u = G \in F^{p,q}_{s+\frac{1}{p}-3}(\Omega), \\ \mathrm{Tr}\, u = f_0 \in B^{p,p}_s(\partial\Omega), \\ \mathrm{Tr}\,(\partial_j u) = f_j \in B^{p,p}_s(\partial\Omega), \ \ 1 \leq j \leq 3, \end{cases} \tag{5.18}$$

where $u \in F^{p,q}_{s+\frac{1}{p}+1}(\Omega)$ and, as before, (5.16) holds.

Proof. Let $\mathscr{P}_{\Delta^2}$ denote the Poisson integral operator for the bi-Laplacian. That is, given a Whitney array $\dot{f} = (f_0, f_1, ..., f_n)$ on $\partial\Omega$, we let $\mathscr{P}_{\Delta^2}\dot{f}$ stand for the solution u of the boundary value problem (3.18). Then, using Theorem 3.1 and Proposition 3.2, we have that

$$\mathscr{P}_{\Delta^2} : W\!A\big(L^p(\partial\Omega)\big) \longrightarrow B^{p,p\vee 2}_{1+\frac{1}{p}}(\Omega) \cap \mathrm{Ker}\,\Delta^2 \quad \text{isomorphically} \quad \forall\, p \in (2-\varepsilon, \infty). \tag{5.19}$$

Also, by Theorem 3.6 and Proposition 3.4, we have

$$\mathscr{P}_{\Delta^2} : WA\big(L^p_1(\partial\Omega)\big) \longrightarrow B^{p,p\vee 2}_{2+\frac{1}{p}}(\Omega) \cap \operatorname{Ker}\Delta^2 \quad \text{isomorphically} \ \ \forall\, p \in (1, 2+\varepsilon). \qquad (5.20)$$

Thus, via real interpolation (cf. (4.83) and (4.54)), we obtain that

$$\begin{array}{c} \mathscr{P}_{\Delta^2} : WA\big(B^{p,q}_s(\partial\Omega)\big) \longrightarrow B^{p,q}_{1+s+\frac{1}{p}}(\Omega) \cap \operatorname{Ker}\Delta^2 \quad \text{is an isomorphism} \\ \text{whenever} \ \ 2-\varepsilon < p < 2+\varepsilon, \quad 0 < q < \infty, \ \text{and} \ 0 < s < 1. \end{array} \qquad (5.21)$$

This portion of the argument is valid in all space dimensions. Consider next the case when $n = 3$. Given the result in Theorem 5.2 and Proposition 3.4, we have

$$\begin{array}{c} \mathscr{P}_{\Delta^2} : WA\big(B^{p,q}_s(\partial\Omega)\big) \longrightarrow B^{p,q}_{1+s+\frac{1}{p}}(\Omega) \cap \operatorname{Ker}\Delta^2 \quad \text{is an isomorphism if} \\ \frac{2}{2+\varepsilon} < p < 1, \quad 0 < q \leq \infty, \quad 1-\varepsilon < s < 1, \quad 2\Big(\frac{1}{p} - 1\Big) < s - 1 + \varepsilon. \end{array} \qquad (5.22)$$

Finally, by Theorem 3.24 and (2.43), (4.31), we also have

$$\mathscr{P}_{\Delta^2} : WA\big(B^{\infty,\infty}_s(\partial\Omega)\big) \longrightarrow B^{\infty,\infty}_{1+s}(\Omega) \cap \operatorname{Ker}\Delta^2 \quad \text{isomorphically if } 0 < s < \varepsilon. \qquad (5.23)$$

Interpolating amongst (5.22), (5.21) and (5.23) by the complex method (cf. (4.82) and Theorem 4.15), we see that

$$\begin{array}{c} \mathscr{P}_{\Delta^2} : WA\big(B^{p,q}_s(\partial\Omega)\big) \longrightarrow B^{p,q}_{1+s+\frac{1}{p}}(\Omega) \cap \operatorname{Ker}\Delta^2 \quad \text{boundedly} \\ \text{if } 0 < q \leq \infty \text{ and } s, p \text{ satisfy one of the conditions in (5.14).} \end{array} \qquad (5.24)$$

Granted Theorem 4.19, the above reasoning shows that the problem (5.15)-(5.16) has a solution satisfying (5.17) whenever s, p, q are as in the second line of (5.24).

The reasoning for (5.18) is similar. The most notable difference is that we specialize (5.24) to the case when $p = q$ and then invoke (4.62), in order to switch from Besov to Triebel-Lizorkin spaces in Ω.

Let us now present an alternative argument for this existence result which has a couple of attractive features. First, this directly yields uniqueness for (5.15) for the full range of indices indicated in the statement of the theorem and, second, it provides a better heuristic understanding of the fact that the region $\{(s, 1/p) \in (0,1)\times(0,1) : \ s, p \text{ satisfy either (I) or (II) in (5.14)}\}$ is symmetric with respect to the point $(1/2, 1/2)$. Specifically, Proposition 4.11 and Proposition 4.10, allow us to rephrase the existence result implicit in (5.24) by saying that

$$\begin{array}{c} \Delta^2 : B^{p,q}_{1+s+\frac{1}{p},z}(\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}-3}(\Omega) \quad \text{is an onto operator} \\ \text{if } 0 < q \leq \infty \text{ and } s, p \text{ satisfy one of the conditions in (5.14).} \end{array} \qquad (5.25)$$

With this in hand, and recalling that the spaces in question form complex interpolation scales, it follows from Theorem 2.10 in [23] that the operator under discussion is actually an isomorphism for the same range of indices, since it is invertible when $p = q = 2$ and $s = 1/2$. Indeed, the fact that

$$\Delta^2 : B^{2,2}_{2,z}(\Omega) \longrightarrow B^{2,2}_{-2}(\Omega) \quad \text{isomorphically} \tag{5.26}$$

is essentially Lax-Milgram's lemma for the bi-Laplacian.

Note that the operator in (5.25) behaves invariantly under duality if one restricts attention to indices $p, q \in (1, \infty)$ −cf. formulas (4.35)-(4.36). This and interpolation (cf. the claim about (4.98)) then show that the maximal region of invertibility (for Δ^2 in (5.25)) within the square $(s, 1/p) \in (0, 1) \times (0, 1)$ is necessarily an open, convex region, which is symmetric with respect to the point $(1/2, 1/2)$.

Returning to the mainstream discussion, we may conclude that Δ^2 maps $B^{p,q}_{1+s+\frac{1}{p},z}(\Omega)$ isomorphically into $B^{p,q}_{s+\frac{1}{p}-3}(\Omega)$ whenever s, p, q are as in (5.14). Once again by virtue of Theorem 4.19 and Propositions 4.11-4.10, this amounts to the well-posedness of (5.15) in the said range of indices.

In the case of Triebel-Lizorkin spaces, we follow the same strategy with one significant difference. Specifically, in (5.25), we shall employ the scale $\{u \in F^{p,q}_{s+1/p+1}(\Omega) : \mathrm{Tr}_{\#} u = 0\}$, indexed by $0 < p, q < \infty$, $(n-1)(1/p-1)_+ < s < 1$, in place of $F^{p,q}_{s+1/p+1,z}(\Omega)$. That the former family of spaces makes up a complex interpolation scale is ensured by Proposition 4.24. The reason we embark on this alternative route is to avoid using Propositions 4.11-4.10 which, in the case of Triebel-Lizorkin spaces, would require imposing the restriction $\min\{1, p\} \le q < \infty$. □

Having disposed of Theorem 5.3, we finally tackle the

Proof of Theorem 1.1. Denote by R_ε the collection of all pairs $(s, 1/p)$ satisfying either I or II in (5.14). The claim we make is that

$$\begin{gathered}\Delta^2 : B^{p,q}_{s+\frac{1}{p}+1,z}(\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}-3}(\Omega) \quad \text{is an isomorphism} \\ \text{whenever } (s, 1/p) \in R_\varepsilon \text{ and } 0 < q < \infty.\end{gathered} \tag{5.27}$$

The proof of this claim is divided into three steps.

`Step 1.` *The operator in the first line of (5.27) is one-to-one, whenever* $(s, 1/p) \in R_\varepsilon$ *and* $0 < q < \infty$. To see this, fix a covering of $\bar{\Omega}$ with finitely many open sets $\{\mathcal{O}_j\}_{1 \le j \le N}$, such that

$$\Omega_j := \mathcal{O}_j \cap \Omega \text{ is a star-like Lipschitz domain, for every } j \in \{1, ..., n\}, \tag{5.28}$$

Also, select a smooth partition of unity

$$\sum_{j=1}^{N}\psi\equiv 1 \text{ near } \bar{\Omega},\quad \psi_j\in C_0^\infty(\mathbb{R}^3),\quad 0\leq\psi_j\leq 1,\quad \operatorname{supp}\psi_j\subset\mathcal{O}_j,\ 1\leq j\leq N. \tag{5.29}$$

Assume next that $(s,1/p)\in R_\varepsilon$, $0<q<\infty$, and that $u\in B^{p,q}_{s+1/p+1,z}(\Omega)$ is such that $\Delta^2 u=0$ in Ω. Then, for every j,

$$\begin{aligned}\Delta^2(\psi_j u) &= (\Delta^2\psi_j)u+2\langle\nabla\Delta\psi_j,\nabla u\rangle+2\Delta^2\psi_j\Delta^2 u\\ &\quad +2\langle\nabla\psi_j,\nabla\Delta u\rangle\in B^{p,q}_{s+1/p-2}(\Omega),\end{aligned} \tag{5.30}$$

so that (by (4.47)),

$$v_j:=(\psi_j u)\Big|_{\Omega_j}\Longrightarrow v_j\in B^{p,q}_{s+1/p+1,z}(\Omega_j)\ \text{ and }\ \Delta^2 v_j\in B^{p,q}_{s+1/p-2}(\Omega_j). \tag{5.31}$$

Furthermore, we claim that

$$\widetilde{v_j}|_\Omega=\psi_j u\ \text{ in }\ \Omega,\quad \text{for every }\ j=1,2,...,N, \tag{5.32}$$

where tilde denotes the extension by zero into $\mathbb{R}^3$. Indeed, the difference $\widetilde{v_j}-\widetilde{\psi_j u}$ is a function in $B^{p,q}_{s+1/p+1}(\mathbb{R}^3)$ whose support is contained in $\partial\Omega_j$. Thus, by Proposition 4.12, this necessarily vanishes, proving (5.32).

Moving on, observe that

$$B^{p,q}_{s+1/p-2}(\Omega_j)\hookrightarrow B^{p_o,q_o}_{s_o+1/p_o-3}(\Omega_j)\ \text{whenever}\ (s_o,1/p_o)\in\Delta_{s,p}\ \text{and}\ 0<q_o<\infty, \tag{5.33}$$

where

$$\begin{aligned}&\Delta_{s,p}\ \text{is the open triangular region with vertices at}\\ &A_1=(0,\tfrac{1}{p}+s+1),\quad A_2=(0,\tfrac{1}{p}-\tfrac{s}{2}-\tfrac{1}{2}),\quad A_3=(s+1,\tfrac{1}{p}).\end{aligned} \tag{5.34}$$

Note that A_1A_2 is vertical and the slopes of A_1A_3, A_2A_3 are -1 and $\frac{1}{2}$, respectively. Based on Proposition 4.26, (5.31)-(5.34), the fact that (5.27) holds with Ω replaced by the star-like Lipschitz domain Ω_j (cf. Theorem 5.3) and the claim about (4.98), we may then infer that

$$v_j\in B^{p_o,q_o}_{s_o+1/p_o+1,z}(\Omega_j)\ \text{whenever}\ (s_o,1/p_o)\in\Delta_{s,p}\ \text{and}\ 0<q_o<\infty. \tag{5.35}$$

Since this is valid for every j, we then obtain $u=\sum_{j=1}^N\widetilde{v_j}|_\Omega\in B^{p_o,q_o}_{s_o+1/p_o+1,z}(\Omega)$ whenever $(s_o,1/p_o)\in\Delta_{s,p}$ and $0<q_o<\infty$. That is, u exhibits better smoothing properties on the

Besov scale than originally assumed. Iterating this scheme finitely many times eventually yields $u \in B^{2,2}_{2,z}(\Omega)$. Given that u is biharmonic, this forces $u = 0$ by (5.26).

Step 2. *The operator in the first line of (5.27) has closed range, whenever* $(s, 1/p) \in R_\varepsilon$ *and* $0 < q < \infty$. To justify this, retain some of the notation introduced in the previous step. Then for every $u \in B^{p,q}_{s+\frac{1}{p}+1,z}(\Omega)$, using (5.32), (5.31), the theory developed in star-like Lipschitz domains, and the boundedness of the restriction and extension by zero operators, we have

$$\begin{aligned}
\|u\|_{B^{p,q}_{s+\frac{1}{p}+1,z}(\Omega)} &\leq C\sum_{j=1}^{N}\|\widetilde{v_j}|_{\Omega}\|_{B^{p,q}_{s+\frac{1}{p}+1,z}(\Omega)} \leq C\sum_{j=1}^{N}\|\widetilde{v_j}\|_{B^{p,q}_{s+\frac{1}{p}+1}(\mathbb{R}^3)} \\
&\leq C\sum_{j=1}^{N}\|v_j\|_{B^{p,q}_{s+\frac{1}{p}+1,z}(\Omega_j)} \leq C\sum_{j=1}^{N}\|\Delta^2 v_j\|_{B^{p,q}_{s+\frac{1}{p}-3}(\Omega_j)} \\
&\leq C\sum_{j=1}^{N}\|(\psi_j\Delta^2 u)|_{\Omega_j}\|_{B^{p,q}_{s+\frac{1}{p}-3}(\Omega_j)} + C\sum_{j=1}^{N}\|u|_{\Omega_j}\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega_j)} \\
&\leq C\|\Delta^2 u\|_{B^{p,q}_{s+\frac{1}{p}-3}(\Omega)} + C\|u\|_{B^{p,q}_{s+\frac{1}{p}-2}(\Omega)}. \qquad (5.36)
\end{aligned}$$

Since the embedding $B^{p,q}_{s+\frac{1}{p}-2}(\Omega) \hookrightarrow B^{p,q}_{s+\frac{1}{p}-3}(\Omega)$ is compact, the above estimate shows that, modulo a compact operator, Δ^2 in the first line of (5.27) is bounded from below. In turn, this ensures that the operator in question has closed range (in the context of Banach spaces this is well-known; for an extension to the case of quasi-Banach spaces – which is really what is needed here – see the discussion in the appendix of [39]).

Step 3. *The operator in the first line of (5.27) is invertible, whenever* $(s, 1/p) \in R_\varepsilon$ *and* $0 < q < \infty$. From Step 1 and Step 2 we know that the operator under discussion is an isomorphic embedding for all values of parameters as in the second line of (5.27). Also, from (5.26), this is actually an isomorphism when $p = q = 2$ and $s = 1/2$. Granted that the spaces in question are complex interpolation scales, the global invertibility result from Theorem 2.10 in [23] then proves that Δ^2 in (5.27) is invertible.

This concludes the proof of the claim made at the beginning of the proof (cf. (5.27)). In turn, using the trace/extension results proved in § 4.4, this translates into the well-posedness of (1.2) for the indicated range of indices.

Consider next (1.5). We start by specializing (5.27) to the case $p = q$, thus obtaining

$$\Delta^2 : F^{p,p}_{s+\frac{1}{p}+1,z}(\Omega) \longrightarrow F^{p,p}_{s+\frac{1}{p}-3}(\Omega) \quad \text{is an isomorphism if } (s, 1/p) \in R_\varepsilon. \qquad (5.37)$$

We now make the claim that

$$\begin{gathered}
\mathrm{Tr}_{\#} : \mathrm{Ker}\,\Delta^2 \cap F^{p,p}_{s+\frac{1}{p}+1}(\Omega) \longrightarrow W\!A\big(B^{p,p}_s(\partial\Omega)\big) \\
\text{is an isomorphism whenever } (s, 1/p) \in R_\varepsilon.
\end{gathered} \qquad (5.38)$$

Indeed, its null-space is precisely

$$\mathrm{Ker}\,\Delta^2\cap\{u\in F^{p,p}_{s+\frac{1}{p}+1}(\Omega):\ \mathrm{Tr}_{\#}u=0\}=\mathrm{Ker}\,\Delta^2\cap F^{p,p}_{s+\frac{1}{p}+1,z}(\Omega)=0, \tag{5.39}$$

by Propositions 4.10-4.11 and (5.37). To see that $\mathrm{Tr}_{\#}$ in (5.38) is also onto, fix p and s such that $(s,1/p)\in\mathcal{R}_\varepsilon$ and let $\dot f\in \mathit{WA}\big(B^{p,p}_s(\partial\Omega)\big)$ be arbitrary. Then, by Theorem 4.19, $v:=\mathrm{Ex}_{\#}(\dot f)$ satisfies $v\in F^{p,p}_{s+1/p+1}(\Omega)$ and $\mathrm{Tr}_{\#}v=\dot f$. Thus, $G:=\Delta^2 v\in F^{p,p}_{s+1/p-3}(\Omega)$ and so

$$(\Delta^2)^{-1}G\in F^{p,p}_{s+\frac{1}{p}+1,z}(\Omega)=\big\{w\in F^{p,q}_{s+\frac{1}{p}+1}(\Omega):\ \mathrm{Tr}_{\#}w=0\big\}, \tag{5.40}$$

once again by Propositions 4.10-4.11. If we set $u:=v-(\Delta^2)^{-1}G$ then

$$u\in\mathrm{Ker}\,\Delta^2\cap F^{p,p}_{s+\frac{1}{p}+1}(\Omega)\quad\text{and}\quad \mathrm{Tr}_{\#}u=\mathrm{Tr}_{\#}v-\mathrm{Tr}_{\#}\big((\Delta^2)^{-1}G\big)=\dot f, \tag{5.41}$$

which shows that $\mathrm{Tr}_{\#}$ in (5.38) is onto and, hence, concludes the proof of (5.38). In fact, by virtue of Theorem 4.16, (5.38) self-improves to

$$\begin{array}{c}\mathrm{Tr}_{\#}:\mathrm{Ker}\,\Delta^2\cap F^{p,q}_{s+\frac{1}{p}+1}(\Omega)\longrightarrow \mathit{WA}\big(B^{p,p}_s(\partial\Omega)\big)\\ \text{isomorphically, if } 0<q<\infty \text{ and } (s,1/p)\in\mathcal{R}_\varepsilon.\end{array} \tag{5.42}$$

In turn, (5.42) readily entails the well-posedness of the problem (1.2) for the range of indices as in the second line of (5.42). □

We wish to point out that the *exterior* versions of (5.15), (5.18) can be also shown to be well-posed under the same assumptions on the indices as in Theorem 5.3. To illustrate this idea, we elaborate briefly on the exterior version of (5.15) written with $G=0$, in order to minimize the technicalities. Specifically, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, define $B^{p,q}_s(\bar\Omega_-;loc)$ as the subspace of distributions u in Ω_- with the property that $u|_{B(0,R)\setminus\bar\Omega}\in B^{p,q}_s(B(0,R)\setminus\bar\Omega)$ for every $R>0$. Then the exterior boundary value problem

$$\left\{\begin{array}{l}\Delta^2u=0\quad\text{in}\quad\Omega_-,\\ \mathrm{Tr}\,u=f_0\in B^{p,q}_s(\partial\Omega),\\ \mathrm{Tr}\,(\partial_j u)=f_j\in B^{p,q}_s(\partial\Omega),\ 1\le j\le 3,\\ u\in B^{p,q}_{s+\frac{1}{p}+1}(\bar\Omega_-;loc),\\ u\text{ decays as in (2.108)},\end{array}\right. \tag{5.43}$$

is well-posed whenever either of the conditions in (5.14) is satisfied. Indeed, this is a consequence of Theorem 1.1 and the results in § 2.4. In fact, a similar result is also valid on the Triebel-Lizorkin scale. For a related version of (5.43), with a different type of decay condition, see the discussion at the end of § 5.2.

We conclude this subsection with the

Proof of Corollary 1.5. For $\varepsilon > 0$ fixed, let $\mathcal{H}_\varepsilon$ be the pentagonal region depicted in Figure 1 (in the statement of Corollary 1.3). Since

$$p \in (2, \infty) \quad \text{and} \quad \tfrac{\alpha+2}{3} = \tfrac{1}{p} \quad \Longrightarrow \quad (\alpha, 1/p) \in \mathcal{H}_\varepsilon, \tag{5.44}$$

by Theorem 1.3 and Theorem 1 on p. 32 of [48], we obtain

$$\nabla \mathbf{G} : B^{p,1}_{\frac{3}{p}-3}(\Omega) \longrightarrow B^{p,1}_{\frac{3}{p}}(\Omega) \hookrightarrow C(\bar{\Omega}) \quad \text{for each} \quad p \in (2, \infty). \tag{5.45}$$

The fact that one can actually allow $p \in (0, \infty)$ is then a consequence of the monotonicity of the scale $B^{p,1}_{3/p-3}(\Omega)$. □

5.2 The biharmonic single layer

We begin with some functional analytic preliminaries. Let $\mathscr{X}$ be a Banach space, and denote by $\mathscr{X}^*$ its (topological) dual and by $\sigma(\mathscr{X}, \mathscr{X}^*)$ the weak topology on $\mathscr{X}$. Also, denote by $\sigma(\mathscr{X}^*, \mathscr{X})$ the weak-$*$ topology on $\mathscr{X}^*$. It is well-known (indeed, a standard consequence of the Hahn-Banach Theorem) that a linear subspace $V \subseteq \mathscr{X}$, V is (norm) closed if and only if V is $\sigma(\mathscr{X}, \mathscr{X}^*)$-closed.

Given two arbitrary linear subspaces $V \subseteq \mathscr{X}$ and $W \subseteq \mathscr{X}^*$ define the annihilators

$$V^\perp := \{\Lambda \in \mathscr{X}^* : \Lambda(x) = 0, \ \forall\, x \in V\}, \tag{5.46}$$

$$W^\perp := \{x \in \mathscr{X} : \Lambda(x) = 0, \ \forall\, \Lambda \in W\}. \tag{5.47}$$

Then $V^\perp \subseteq \mathscr{X}^*$, $W^\perp \subseteq \mathscr{X}$ are (norm) closed subspaces, and (cf. Theorem 4.7 on p. 96 in [47])

$$(V^\perp)^\perp \text{ is the norm-closure of } V \text{ in } \mathscr{X}, \text{ and} \tag{5.48}$$

$$(W^\perp)^\perp \text{ is the } \sigma(\mathscr{X}^*, \mathscr{X})\text{-closure of } W \text{ in } \mathscr{X}^*. \tag{5.49}$$

Denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between $\mathscr{X}$ and $\mathscr{X}^*$.

Proposition 5.4. *Assume that $\mathscr{X}$ is a Banach space and that $S : \mathscr{X}^* \to \mathscr{X}$ is a linear, bounded operator, with closed range, and which satisfies the following additional property. There exists a finite dimensional subspace V of $\mathscr{X}$ such that*

$$\Lambda \in V^\perp \ \textit{and} \ \langle S\Lambda, \Lambda \rangle = 0 \implies \Lambda = 0. \tag{5.50}$$

Then S is Fredholm, with index zero, and

$$S : V^{\perp} \longrightarrow \mathscr{X}/V \tag{5.51}$$

is an isomorphism. Furthermore, denoting by $\operatorname{Ker} S$ *the kernel of the operator* $S : \mathscr{X}^* \to \mathscr{X}$*, there holds*

$$\dim(\operatorname{Ker} S) \leq \dim V. \tag{5.52}$$

Proof. We claim that

$$S(V^{\perp}) \oplus V = \mathscr{X}. \tag{5.53}$$

To justify this, note that if $x \in V \cap S(V^{\perp})$ then there exists $\Lambda \in V^{\perp}$ such that $x = S\Lambda \in V$. This forces $\langle S\Lambda, \Lambda \rangle = \langle x, \Lambda \rangle = 0$, hence $\Lambda = 0$ by (5.50). Thus the sum in (5.53) is direct.

Next, observe that (5.50) entails $V^{\perp} \cap \big(S(V^{\perp})\big)^{\perp} = \{0\}$. Using this we may then write $(S(V^{\perp}) \oplus V)^{\perp} \subseteq V^{\perp} \cap (S(V^{\perp}))^{\perp} = \{0\}$ which, given (5.48), proves that $S(V^{\perp}) \oplus V$ is (norm) dense in $\mathscr{X}$. Therefore, we are left with showing that $S(V^{\perp}) \oplus V$ is (norm) closed in $\mathscr{X}$. In turn, since a finite dimensional augmentation of a linear subspace preserves its closedness, this is going to be a consequence of the fact that $S(V^{\perp})$ is (norm) closed in $\mathscr{X}$. As far as the latter issue is concerned, the fact that the operator $S : \mathscr{X}^*/V^{\perp} \to S\mathscr{X}^*/S(V^{\perp})$ is well-defined, linear and onto implies (cf. also § 4.8 on p. 96 in [47])

$$\dim\,[S\mathscr{X}^*/S(V^{\perp})] \leq \dim\,[\mathscr{X}^*/V^{\perp}] = \dim\,[V^*] = \dim\,[V] < +\infty. \tag{5.54}$$

It follows then that the operator $S : V^{\perp} \to S\mathscr{X}^*$, acting linearly and boundedly between Banach spaces (recall that it is assumed that S has closed range) has a finite codimensional range. Based on this we may then conclude (cf. Lemma 2 on p. 156 in [40]) that $S(V^{\perp})$ is (norm) closed in $S\mathscr{X}^*$ and, hence, in $\mathscr{X}$. This finishes the proof of (5.53).

With (5.53) in hand and further relying on (5.51), it is then easy to show that the operator in (5.51) is an isomorphism. This and standard functional analysis then give that $S : \mathscr{X}^* \to \mathscr{X}$ is Fredholm with index zero.

Consider now (5.52). Take a linear basis $\{v_1, ..., v_d\}$ of V, where $d := \dim V$, and denote by $\{\Lambda_1, ..., \Lambda_d\} \subset \mathscr{X}^*$ its dual system, i.e., $\langle \Lambda_j, v_k \rangle = \delta_{jk}$ for every $j, k \in \{1, ..., d\}$. In particular, $\Lambda_1, ..., \Lambda_d$ are linearly independent. Finally, set

$$W := \operatorname{span}\{\Lambda_1, ..., \Lambda_d\}, \tag{5.55}$$

and introduce the linear operator

$$\pi : \operatorname{Ker} S \longrightarrow W, \qquad \pi(\Lambda) := \sum_{j=1}^{d} \langle \Lambda, v_j \rangle \Lambda_j, \tag{5.56}$$

where $\operatorname{Ker} S \subset \mathscr{X}^*$ is the kernel of the operator $S : \mathscr{X}^* \to \mathscr{X}$. Then π in (5.56) is one-to-one. Indeed, let $\Lambda \in \mathscr{X}^*$ be such that $S\Lambda = 0$ and $\pi(\Lambda) = 0$. Since $\pi(\Lambda) = 0$ is equivalent to the membership of Λ in $V^\perp$, and the operator S in (5.51) is an isomorphism, we may conclude that $\Lambda = 0$, as wanted. The fact that the operator π in (5.56) is one-to-one now readily implies (5.52), since $\dim W = d$. □

Let us now introduce a suitable concept of single layer operator associated to the bi-Laplacian. Following the recipe in [36], given a bounded Lipschitz domain Ω in $\mathbb{R}^n$, we set

$$\begin{aligned}(\dot{\mathcal{S}}\Lambda)(X) &:= \Big\langle \Lambda\,,\, \mathrm{Tr}_{\#}\big[B(X-\cdot)\big]\Big\rangle \\ &= \Big\langle \Lambda(Y)\,,\, \big(B(X-Y), \nabla_Y[B(X-Y)]\big)\Big\rangle, \qquad X \in \mathbb{R}^n \setminus \partial\Omega, \qquad (5.57)\end{aligned}$$

where Λ is a functional on the space of Whitney arrays (exhibiting a certain amount of smoothness −more details on this later). The reader is advised that this is a different version than the one studied in the (mostly engineering) literature. Indeed, the "standard" biharmonic single layer reads

$$\begin{aligned}\mathcal{S}_{\rm st}(F,G)(X) &:= \int_{\partial\Omega} B(X-Y)F(Y)\,d\sigma(Y) \\ &\quad + \int_{\partial\Omega} \partial_{\nu(Y)}[B(X-Y)]G(Y)\,d\sigma(Y), \quad X \in \mathbb{R}^n \setminus \partial\Omega, \qquad (5.58)\end{aligned}$$

where F, G are functions defined on the boundary of the Lipschitz domain $\Omega \subset \mathbb{R}^n$. We wish to briefly elaborate on the relationship between the single layer in (5.57) and the one in (5.58) and explain why, in the current context, (5.57) is more suitable for the purposes we have in mind.

While dealing with the inhomogeneous Dirichlet problem for the bi-Laplacian $\Delta^2 u = f$, $u \in \mathring{W}^{2,2}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{2,2}(\Omega)$), it is useful to have an integral operator which maps boundary functions into the energy space $W^{2,2}(\Omega)$. Hence, one desirable property for the operator (5.58) is that it maps into $W^{2,2}(\Omega)$. Homogeneity considerations then dictate that

$$\mathcal{S}_{\rm st} : B^{2,2}_{-3/2}(\partial\Omega) \oplus B^{2,2}_{-1/2}(\partial\Omega) \longrightarrow W^{2,2}(\Omega), \qquad \text{boundedly.} \qquad (5.59)$$

The problem with (5.59) is that, on the boundary of a Lipschitz domain, the Besov space $B^{2,2}_{-3/2}(\partial\Omega)$ is not well-defined. Thus, if one insists that arbitrary Lipschitz domains are considered, it is necessary to reconfigure the operator (5.58) as to avoid the aforementioned problem. To this end, given two functions F, G on $\partial\Omega$, assume that there exist functions $g_0, g_1, ..., g_n$ such that

$$F = g_0 + \partial_{\tau_{jk}}(\nu_k g_j), \qquad G = \nu_j g_j. \qquad (5.60)$$

If that is the case, manipulate the right-hand side of (5.58) as follows

$$\begin{aligned}\mathcal{S}_{\rm st}(F,G)(X) &= \int_{\partial\Omega} B(X-Y)(g_0+\partial_{\tau_{jk}}(\nu_k g_j))(Y)\,d\sigma(Y)\\ &\quad+\int_{\partial\Omega}\partial_{\nu(Y)}[B(X-Y)](\nu_j g_j)(Y)\,d\sigma(Y) \qquad (5.61)\\ &= \int_{\partial\Omega} B(X-Y)g_0(Y)\,d\sigma(Y)+\int_{\partial\Omega}\partial_j^Y[B(X-Y)]g_j(Y)\,d\sigma(Y),\end{aligned}$$

after integrating by parts on the boundary. Hence,

$$\begin{aligned}\mathcal{S}_{\rm st}(F,G)(X) &= \Big\langle \Lambda_{g_0,\dots,g_n}(Y)\,,\,\big(B(X-Y),\nabla_Y[B(X-Y)]\big)\Big\rangle\\ &= (\dot{\mathcal{S}}\Lambda_{g_0,\dots,g_n})(X), \qquad (5.62)\end{aligned}$$

where $\Lambda_{g_0,\dots,g_n}$ is the functional acting on a Whitney array $\dot f=(f_0,f_1,\dots,f_n)$ according to

$$\langle\Lambda_{g_0,\dots,g_n},\dot f\rangle := \sum_{j=0}^n\int_{\partial\Omega} f_j g_j\,d\sigma. \qquad (5.63)$$

In summary:

given F,G, then $\mathcal{S}_{\rm st}(F,G)=\dot{\mathcal{S}}\Lambda_{g_0,\dots,g_n}$ provided $\Lambda_{g_0,\dots,g_n}$ is a functional on Whitney arrays build out of $g_0,\dots,g_n$ as in (5.63) and, further, the g_j's are related to F and G as in (5.60). (5.64)

The question now arises:

what functions F,G can be represented as in (5.60)? (5.65)

To answer this, set

$$\Phi(\Lambda_{g_0,g_1,\dots,g_n}) := (F,G) \quad \text{where } F,G \text{ are as in (5.60).} \qquad (5.66)$$

We claim that this mapping induces a well-defined, bounded linear operator

$$\Phi : W\!A\big(L^{p'}(\partial\Omega)\big)^* \longrightarrow L^p_{-1}(\partial\Omega)\oplus L^p(\partial\Omega), \qquad (5.67)$$

in the following sense. By the Hahn-Banach Theorem, any functional $\Lambda\in W\!A\big(L^{p'}(\partial\Omega)\big)^*$ is as in (5.63), for some not necessarily unique $(n+1)$-tuple of functions $g_0,\dots,g_n\in L^p(\partial\Omega)$.

Then the definition (5.66) is unambiguous, in the sense the outcome is unaffected by the particular choice of g_j's as above. A moment's reflection shows that this is equivalent to proving that, given $g_0, ..., g_n \in L^p(\partial\Omega)$,

$$\sum_{j=0}^{n} \int_{\partial\Omega} f_j g_j \, d\sigma = 0 \;\text{ for all }\; \dot{f} \in WA\big(L^{p'}(\partial\Omega)\big) \implies F, G \text{ in (5.60) vanish.} \tag{5.68}$$

To see this, we note that

$$\langle \Lambda_{g_0,\dots,g_n}, Q(F', G')\rangle = \langle (F,G), (F',G')\rangle \tag{5.69}$$

for every $g_0, ..., g_n \in L^p(\partial\Omega)$, every $(F', G') \in L_1^{p'}(\partial\Omega) \oplus L^{p'}(\partial\Omega)$, and with (F, G) as in (5.60); this is clear from definitions. Since, according to (3.26)-(3.27), $Q(F', G')$ is a Whitney array in $WA\big(L^{p'}(\partial\Omega)\big)$, (5.68) follows from (5.69).

Formula (5.69) also proves that, in the context of (5.67), the operator Φ is the adjoint of Q in (3.27). Since, as mentioned in the remark following the proof of Proposition 3.2, Q in (3.27) is an isomorphism, it follows that Φ in (5.67) is an isomorphism as well. As a consequence, in relation to question (5.65), we infer that any $(F, G) \in L_{-1}^p(\partial\Omega) \oplus L^p(\partial\Omega)$ can be represented as $\Phi(\Lambda_{g_0,\dots,g_n})$ for some $g_0, ..., g_n \in L^p(\partial\Omega)$. Hence,

$$\mathcal{S}_{\rm st} = \dot{\mathcal{S}} \circ \Phi^{-1} \qquad \text{as operators on } \; L_{-1}^p(\partial\Omega) \oplus L^p(\partial\Omega). \tag{5.70}$$

That is, the two biharmonic single layers (5.57), (5.58) agree up to an isomorphism of $L_{-1}^p(\partial\Omega) \oplus L^p(\partial\Omega)$ onto $WA\big(L^{p'}(\partial\Omega)\big)^*$. However, as opposed to (5.59) which requires Ω to be smoother than Lipschitz, for the operator (5.57) we have that

$$\dot{\mathcal{S}} : WA\big(B_{1/2}^{2,2}(\partial\Omega)\big)^* \longrightarrow W^{2,2}(\Omega) \tag{5.71}$$

is well-defined and bounded (the different behavior is due to the fact that, generally speaking, Φ does not map onto $B_{-3/2}^{2,2}(\partial\Omega) \oplus B_{-1/2}^{2,2}(\partial\Omega)$). Operators such as (5.57) have been studied in [36], in more generality. Here we record a couple of results established in [36] which are going to play a significant role in the present work. First,

$$\dot{\mathcal{S}} : WA\big(B_{1-s}^{p,q}(\partial\Omega)\big)^* \longrightarrow B_{1+s+\frac{1}{p'}}^{p',q'}(\Omega) \tag{5.72}$$

is well-defined and bounded whenever $1 < p, q < \infty$, and $0 < s < 1$ (as is customary, $1/p + 1/p' = 1/q + 1/q' = 1$). Furthermore, similar results are valid on the Triebel-Lizorkin scale, and for the exterior version of (5.72). In addition, with $\mathrm{Tr}_{\#}$ as in Theorem 4.19, the boundary biharmonic single layer

$$\dot{S} := \mathrm{Tr}_{\#} \circ \dot{\mathcal{S}} : WA\big(B_{1-s}^{p,q}(\partial\Omega)\big)^* \longrightarrow WA\big(B_s^{p',q'}(\partial\Omega)\big) \tag{5.73}$$

is also well-defined and bounded if $1 < p, q < \infty$ and $0 < s < 1$.

Moving on, for a sufficiently nice function u in Ω, define

$$\partial_\nu^{\#,+} u := \Big((\partial_\nu \Delta u)\Big|_{\partial\Omega}, (\partial_\nu \partial_1 u)\Big|_{\partial\Omega}, ..., (\partial_\nu \partial_n u)\Big|_{\partial\Omega} \Big). \tag{5.74}$$

Similarly we introduce $\partial_\nu^{\#,-} u$ for sufficiently nice functions u defined in Ω_-. Formula (5.74) is designed such that if u and v are two reasonably behaved functions in Ω, the following Green formula holds:

$$\begin{aligned} &\sum_{j,k=1}^{n} \int_\Omega (\partial_j \partial_k u)(X)(\partial_j \partial_k v)(X)\, dX \qquad (5.75) \\ &\qquad = - \int_{\partial\Omega} \Big\langle \partial_\nu^{\#,+} u(X), \mathrm{Tr}_{\#} v(X) \Big\rangle \, d\sigma(X) + \int_\Omega (\Delta^2 u)(X) v(X)\, dX. \end{aligned}$$

In turn, the identity (5.75) suggests defining

$$\partial_\nu^{\#,+} : B^{p,q}_{s+\frac{1}{p}+1}(\Omega) \cap \mathrm{Ker}\, \Delta^2 \longrightarrow WA\big(B^{p',q'}_{1-s}(\partial\Omega)\big)^* \tag{5.76}$$

by setting

$$\Big\langle \partial_\nu^{\#,+} u, \phi \Big\rangle := - \sum_{j,k=1}^{n} \int_\Omega (\partial_j \partial_k u)(X)(\partial_j \partial_k \Phi)(X)\, dX \tag{5.77}$$

for every $\phi \in WA\big(B^{p',q'}_{1-s}(\partial\Omega)\big)$, where $\Phi \in B^{p',q'}_{2-s+\frac{1}{p'}}(\Omega)$ is any function with the property that $\mathrm{Tr}_{\#}\Phi = \phi$. We define

$$\partial_\nu^{\#,-} : B^{p,q}_{s+\frac{1}{p}+1}(\Omega_-; loc) \cap \mathrm{Ker}\, \Delta^2 \longrightarrow WA\big(B^{p',q'}_{1-s}(\partial\Omega)\big)^* \tag{5.78}$$

analogously to (5.77), with the convention that the extension Φ of ϕ is taken to have bounded support.

Proposition 5.5. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and fix three indices $1 < p, q < \infty$ and $0 < s < 1$. Then the above definition of $\partial_\nu^{\#,+}$ is meaningful, and the operator in (5.76) is bounded. Furthermore, similar considerations apply to*

$$\partial_\nu^{\#,+} : F^{p,q}_{s+\frac{1}{p}+1}(\Omega) \cap \mathrm{Ker}\, \Delta^2 \longrightarrow WA\big(B^{p',p'}_{1-s}(\partial\Omega)\big)^*, \tag{5.79}$$

and to the exterior versions.

Proof. In the left-hand side of (5.77), $\langle\cdot,\cdot\rangle$ denotes the duality pairing between functionals in $WA\big(B^{p',q'}_{1-s}(\partial\Omega)\big)^*$ and elements of $WA\big(B^{p',q'}_{1-s}(\partial\Omega)\big)$, while in the right-hand side of (5.77), the integral is interpreted as the duality pairing between functionals in $(B^{p',q'}_{-s+1/p'}(\Omega))^* = B^{p,q}_{s+1/p-1}(\Omega)$ and functions in $B^{p',q'}_{-s+1/p'}(\Omega)$. That the latter is meaningful is ensured by (4.35)-(4.36) and (4.39).

The fact that the right-hand side of (5.77) is independent of the choice of the function $\Phi \in B^{p',q'}_{2-s+\frac{1}{p'}}(\Omega)$ satisfying $\mathrm{Tr}_{\#}\Phi = \phi$ is a consequence of Propositions 4.10-4.11. The boundedness of the operator in (5.76) is then implicit in the above arguments. Similar considerations apply to the conormal derivative on Triebel-Lizorkin spaces and to its exterior version. □

The proposition recorded below, dealing with the boundary behavior of the harmonic single layer, has been established in [36]. To state it, denote by $\mathrm{Tr}^{\pm}_{\#}$ the boundary traces, taken from within $\Omega_{\pm}$, as in Theorem 4.19.

Proposition 5.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $1 < p, q < \infty$ and $0 < s < 1$. Also, let $p', q' \in (1,\infty)$ be such that $1/p + 1/p' = 1/q + 1/q' = 1$. Then*

$$\partial^{\#,-}_{\nu}\dot{\mathcal{S}}\Lambda - \partial^{\#,+}_{\nu}\dot{\mathcal{S}}\Lambda = \Lambda, \tag{5.80}$$

$$\mathrm{Tr}^{+}_{\#}\dot{\mathcal{S}}\Lambda = \mathrm{Tr}^{-}_{\#}\dot{\mathcal{S}}\Lambda, \tag{5.81}$$

for each $\Lambda \in WA\big(B^{p',q'}_{s}(\partial\Omega)\big)^*$.

For each $k \in \mathbb{N}_0$ we denote by $\mathcal{P}_k$ the set of polynomials in $\mathbb{R}^n$ of degree less than or equal to k, and make the convention that $\mathcal{P}_{-1} := \{0\}$. Also, recall that $\eta(X) \equiv X$ for $X \in \mathbb{R}^n$.

Proposition 5.7. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, where $n \geq 3$, $n \neq 4$. Let $\alpha \in \mathbb{N}^n_0$, $k \in \mathbb{N} \cup \{0,-1\}$, and fix $1 < p, q < \infty$, $0 < s < 1$. Then for each functional $\Lambda \in WA\big(B^{p,q}_{s}(\partial\Omega)\big)^*$ there holds*

$$\partial^\alpha\dot{\mathcal{S}}\Lambda(X) = \sum_{|\beta|\leq k} \tfrac{(-1)^{|\beta|}}{\beta!}(\partial^{\alpha+\beta}B)(X)\langle\Lambda, \mathrm{Tr}_{\#}(\eta^\beta)\rangle + O(|X|^{3-|\alpha|-n-k}) \quad \text{as} \quad X \to \infty \tag{5.82}$$

(where the sum is void if $k = -1$). If, in addition,

$$\langle\Lambda, \mathrm{Tr}_{\#}\, p\rangle = 0 \qquad \forall\, p \in \mathcal{P}_k, \tag{5.83}$$

(which is a void condition if $k = -1$) then

$$\partial^\alpha\dot{\mathcal{S}}\Lambda(X) = O(|X|^{3-|\alpha|-n-k}) \quad \text{as} \quad X \to \infty. \tag{5.84}$$

Proof. The case $k=-1$ is clear. Fix $\alpha\in\mathbb{N}^n$, $k\in\mathbb{N}_0$ and $\Lambda\in \mathit{WA}\big(B^{p,q}_s(\partial\Omega)\big)^*$. Then, based on (5.57), for $X\in\mathbb{R}^n\setminus\partial\Omega$ we may write

$$\partial^\alpha\dot{\mathcal{S}}\Lambda(X)=\langle\Lambda,\mathrm{Tr}_\#\big((\partial^\alpha B)(X-\cdot)-P^{k,\alpha}_X\big)\rangle+\langle\dot\Lambda,\mathrm{Tr}_\#(P^{k,\alpha}_X)\rangle \tag{5.85}$$

where $P^{k,\alpha}_X$ is the Taylor polynomial of degree k for the function $\partial^\alpha B(X-\cdot)$ about 0. That is, generally speaking,

$$P^{\ell,\alpha}_X(Y):=\sum_{|\beta|\leq\ell}\tfrac{(-1)^{|\beta|}}{\beta!}Y^\beta(\partial^{\alpha+\beta}B)(X) \tag{5.86}$$

if $\ell\geq 0$ and $P^{\ell,\alpha}_X(Y)\equiv 0$ if $\ell<0$. Thus,

$$\langle\Lambda,\mathrm{Tr}_\#(P^{k,\alpha}_X)\rangle=\sum_{|\beta|\leq k}\tfrac{(-1)^{|\beta|}}{\beta!}(\partial^{\alpha+\beta}B)(X)\langle\Lambda,\mathrm{Tr}_\#(\eta^\beta)\rangle. \tag{5.87}$$

It is straightforward to check that for every multi-index γ (of any length) we have

$$\partial^\gamma_Y P^{k,\alpha}_X(Y)=(-1)^{|\gamma|}P^{k-|\gamma|,\alpha+\gamma}_X(Y). \tag{5.88}$$

Therefore, for every γ,

$$\begin{aligned}\Big|\partial^\gamma_Y\big[(\partial^\alpha B)(X-Y)-P^{k,\alpha}_X(Y)\big]\Big| &= |(\partial^{\alpha+\gamma}B)(X-Y)-P^{k-|\gamma|,\alpha+\gamma}_X(Y)|\\ &\leq C\sup_{\substack{Z\in[0,Y]\\ |\delta|=k+|\alpha|+1}}|(\partial^\delta B)(X-Z)||Y|^{k+1-|\gamma|}\end{aligned} \tag{5.89}$$

by (5.88) and Taylor's theorem. Since $\nabla^{k+|\alpha|+1}B(X)=O(|X|^{3-|\alpha|-n-k})$ as $X\to\infty$, (5.89) gives that $\|\mathrm{Tr}_\#\big((\partial^\alpha B)(X-\cdot)-P^{k,\alpha}_X\big)\|_{\mathit{WA}(B^{p,q}_s(\partial\Omega))}=O(|X|^{3-|\alpha|-n-k})$ as $X\to\infty$. Consequently, $\langle\Lambda,\mathrm{Tr}_\#\big((\partial^\alpha B)(X-\cdot)-P^{k,\alpha}_X\big)\rangle=O(|X|^{3-|\alpha|-n-k})$. Now (5.82) follows from this and (5.87), while (5.84) is immediate from (5.82). □

Proposition 5.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then, there exists $\varepsilon=\varepsilon(\Omega)>0$ such that for each $1<q<\infty$ and each $(s,1/p)\in(0,1)\times(0,1)$ satisfying one of the conditions in (1.1) there holds*

$$\dot S:\mathit{WA}\big(B^{p,q}_{1-s}(\partial\Omega)\big)^*\longrightarrow \mathit{WA}\big(B^{p',q'}_s(\partial\Omega)\big)\quad\textit{is Fredholm,} \tag{5.90}$$

where $1/p+1/p'=1/q+1/q'=1$.

Proof. Given $\Omega\subset\mathbb{R}^3$, bounded Lipschitz domain, we let ε be as in the statement of Theorem 1.1. Fix next $1<p,q<\infty$ and $s\in(0,1)$ such that one of the conditions in (1.1) holds. From Proposition 5.5 and (5.72) we know that there exists $C>0$ such that

$$\|\partial_\nu^{\#,+}\dot{\mathcal{S}}\Lambda\|_{WA(B^{p',q'}_{1-s}(\partial\Omega))^*}\leq C\|\dot{\mathcal{S}}\Lambda\|_{B^{p,q}_{s+\frac{1}{p}+1}(\Omega)},\tag{5.91}$$

uniformly for $\Lambda\in WA\big(B^{p',q'}_{1-s}(\partial\Omega)\big)^*$. Next, fix $R>0$ such that $\Omega\subset B_R(0)$ and pick a function $\xi\in C_0^\infty(B_R(0))$, with $\xi\equiv1$ near $\bar\Omega$. Then, much as before (working with $B_R(0)\setminus\bar\Omega$ in place of Ω), we have

$$\|\partial_\nu^{\#,-}\dot{\mathcal{S}}\Lambda\|_{WA(B^{p',q'}_{1-s}(\partial\Omega))^*}\leq C\|\dot{\mathcal{S}}\Lambda\|_{B^{p,q}_{s+\frac{1}{p}+1}(B_R(0)\setminus\bar\Omega)},\tag{5.92}$$

uniformly for $\Lambda\in WA\big(B^{p',q'}_{1-s}(\partial\Omega)\big)^*$. Then, using the jump relations (5.80), (5.91) and (5.92)

$$\begin{aligned}\|\Lambda\|_{WA(B^{p',q'}_{1-s}(\partial\Omega))^*} &\leq \|\partial_\nu^{\#,+}\dot{\mathcal{S}}\Lambda\|_{WA(B^{p',q'}_{1-s}(\partial\Omega))^*}+\|\partial_\nu^{\#,-}\dot{\mathcal{S}}\Lambda\|_{WA(B^{p',q'}_{1-s}(\partial\Omega))^*}\\ &\leq C\|\dot{\mathcal{S}}\Lambda\|_{B^{p,q}_{s+\frac{1}{p}+1}(\Omega)}+C\|\dot{\mathcal{S}}\Lambda\|_{B^{p,q}_{s+\frac{1}{p}+1}(B_R(0)\setminus\bar\Omega)}.\end{aligned}\tag{5.93}$$

Going further, using this, (5.73) (with the understanding that $\mathrm{Tr}_\#$ can be taken from either side of $\partial\Omega$) and the well-posedness results from Theorem 1.1 we obtain

$$\begin{aligned}\|\Lambda\|_{WA(B^{p',q'}_{1-s}(\partial\Omega))^*} &\leq C\|\dot{S}\Lambda\|_{WA(B^{p,q}_s(\partial\Omega))}+C\|\dot{S}\Lambda\|_{WA(B^{p,q}_s(\partial(B_R(0)\setminus\bar\Omega)))}\\ &\leq C\|\dot{S}\Lambda\|_{WA(B^{p,q}_s(\partial\Omega))}+\|\mathrm{Comp}(\Lambda)\|,\end{aligned}\tag{5.94}$$

where Comp denotes a generic compact operator acting from $WA(B^{p',q'}_{1-s}(\partial\Omega))^*$. This shows that

$$\begin{aligned}&\dot{S}:WA\big(B^{p,q}_{1-s}(\partial\Omega)\big)^*\longrightarrow WA\big(B^{p',q'}_s(\partial\Omega)\big)\quad\text{has closed range}\\ &\text{and finite dimensional kernel, whenever } 1<q<\infty \text{ and}\\ &(s,1/p)\in(0,1)^2 \text{ satisfies one of the conditions in (1.1).}\end{aligned}\tag{5.95}$$

Now, the conclusion of Proposition 5.8 follows from (5.95) by duality and simple functional analytic arguments since $\dot{S}$ in (5.73) is formally self-adjoint and the region of indices s,p,q described in the last line of (5.95) is invariant to duality. □

To state our next result, recall (5.46) and that $\mathcal{P}_1$ stands for the space of affine mappings of the Euclidean space. Also, given a Lipschitz domain Ω, set $\dot{\mathcal{P}}_1:=\{\mathrm{Tr}_\# p:\ p\in\mathcal{P}_1\}$.

Theorem 5.9. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then the operator induced by the biharmonic single layer,*

$$\dot{S} : WA\big(B^{p,q}_{1-s}(\partial\Omega)\big)^* \longrightarrow WA\big(B^{p',q'}_{s}(\partial\Omega)\big) \tag{5.96}$$

is an isomorphism whenever

$$1 < p, q < \infty \text{ and } s \in (0,1) \text{ satisfy one of the conditions in (1.1).} \tag{5.97}$$

Proof. We start by claiming that the operator

$$\dot{S} : \{\Lambda \in WA\big(B^{p,q}_{1-s}(\partial\Omega)\big)^* \,:\, \Lambda \in \dot{\mathcal{P}}_0^{\perp}\} \longrightarrow WA\big(B^{p',q'}_{s}(\partial\Omega)\big)\Big/\dot{\mathcal{P}}_0, \tag{5.98}$$

is an isomorphism whenever (5.97) holds. The fact that this operator is Fredholm for the indicated range of indices is a direct consequence of Proposition 5.8. Then, cf. [23], the index of the operator (5.98) is constant with respect to s, p, q. If we are able to show that

$$\dot{S} : \{\Lambda \in WA\big(B^{2,2}_{1/2}(\partial\Omega)\big)^* \,:\, \Lambda \in \dot{\mathcal{P}}_0^{\perp}\} \longrightarrow WA\big(B^{2,2}_{2}(\partial\Omega)\big)\Big/\dot{\mathcal{P}}_0 \quad \text{is an isomorphism,} \tag{5.99}$$

it is then easy to show (via embeddings and elementary functional analysis) that the claim about the operator (5.98) holds.

We therefore concentrate on proving (5.99). To this end, bring in Proposition 5.4 in which we take $V := \dot{\mathcal{P}}_0$, $\mathscr{X} := WA\big(B^{2,2}_{1/2}(\partial\Omega)\big)$ and $S := \dot{S}$. In this scenario, we claim that the implication (5.50) holds. To show this, let $\Lambda \in \dot{\mathcal{P}}_0^{\perp}$ be such that $\langle \dot{S}\Lambda, \Lambda\rangle = 0$. Also, let $R > 0$ be such that $\bar{\Omega} \subset B_R(0)$. Applying (5.75) for $u = v := \dot{\mathcal{S}}\Lambda$ defined, respectively, in Ω and $B_R(0) \setminus \bar{\Omega}$, and recalling (5.80)-(5.81) we obtain

$$\begin{aligned}
&\sum_{j,k} \int_{\Omega} |\partial_j\partial_k \dot{\mathcal{S}}\Lambda|^2 \, dX + \sum_{j,k} \int_{B_R(0)\setminus\bar{\Omega}} |\partial_j\partial_k \dot{\mathcal{S}}\Lambda|^2 \, dX \\
&\qquad = -\langle \partial_\nu^{\#,+}\dot{\mathcal{S}}\Lambda, \dot{S}\Lambda\rangle + \langle \partial_\nu^{\#,-}\dot{\mathcal{S}}\Lambda, \dot{S}\Lambda\rangle + \int_{|X|=R} \langle \partial_\nu^{\#,-}\dot{\mathcal{S}}\Lambda \,,\, \mathrm{Tr}_{\#}\dot{\mathcal{S}}\Lambda\rangle \, d\sigma \\
&\qquad = \langle \Lambda, \dot{S}\Lambda\rangle + \int_{|X|=R} \langle \partial_\nu^{\#,-}\dot{\mathcal{S}}\Lambda \,,\, \mathrm{Tr}_{\#}\dot{\mathcal{S}}\Lambda\rangle \, d\sigma,
\end{aligned} \tag{5.100}$$

where, in the last expression above, the first duality pairing is considered on $\partial\Omega$, and the pairing under the integral sign is taken in the pointwise sense (in which case, the trace, conormal derivative and surface measure are those associated with $\partial B_R(0)$). In relation to the latter pairing, Proposition 5.7 and (5.74) give that

$$
\begin{aligned}
(\partial_\nu^{\#,-}\dot{\mathcal{S}}\Lambda)(X) &= \Big((\partial_\nu\Delta\dot{\mathcal{S}}\Lambda)(X)\,,\,((\partial_\nu\partial_j\dot{\mathcal{S}}\Lambda)(X))_{j=1,2,3}\Big) \\
&= \Big(O(|X|^{-3}),O(|X|^{-2}),O(|X|^{-2}),O(|X|^{-2})\Big) \quad \text{as} \quad X\to\infty,
\end{aligned}
\tag{5.101}
$$

and

$$
\mathrm{Tr}_{\#}\dot{\mathcal{S}}\Lambda(X) = \Big(O(1),O(|X|^{-1}),O(|X|^{-1}),O(|X|^{-1})\Big) \quad \text{as} \quad X\to\infty, \tag{5.102}
$$

since $\Lambda\in\dot{\mathcal{P}}_0^{\perp}$. Thus, in absolute value, the last integral in (5.100) is $\leq CR^2R^{-3}$, i.e., $O(R^{-1})$ as $R\to\infty$. Passing to limit in (5.100), we then arrive at

$$
\langle\Lambda,\dot{S}\Lambda\rangle = \sum_{j,k}\int_\Omega|\partial_j\partial_k\dot{\mathcal{S}}\Lambda|^2\,dX + \sum_{j,k}\int_{\mathbb{R}^n\setminus\bar{\Omega}}|\partial_j\partial_k\dot{\mathcal{S}}\Lambda|^2\,dX, \qquad \forall\,\Lambda\in\dot{\mathcal{P}}_0^{\perp}. \tag{5.103}
$$

Since we are assuming $\langle\Lambda,\dot{S}\Lambda\rangle=0$, this forces $\partial_j\partial_k\dot{\mathcal{S}}\Lambda\equiv0$ in $\Omega_\pm$. Hence, by (5.80), $\Lambda=\partial_\nu^{\#,-}\dot{\mathcal{S}}\Lambda-\partial_\nu^{\#,+}\dot{\mathcal{S}}\Lambda=0$, as desired. Thus, Proposition 5.4 applies, finishing the proof of the claim about the operator (5.98).

Next, as a consequence of what we have just proved and simple functional analysis, the operator $\dot{S}$ in (5.96) is Fredholm with index zero whenever p,q,s are as in (5.97). Thus, to complete the proof of the theorem, it suffices to show that $\dot{S}$ in (5.96) is one-to-one. With this goal in mind, let $\Lambda\in WA\big(B^{p,q}_{1-s}(\partial\Omega)\big)^*$ be such that $\dot{S}\Lambda=0$ and write $\Lambda=\Lambda_0+c\dot{1}$ where $\Lambda_0\in\dot{\mathcal{P}}_0^{\perp}$ and $c\in\mathbb{R}$. Here, $\dot{1}$ denotes the Whitney array $(1,0,0,0)$; in particular,

$$
\langle\dot{S}\dot{1},\dot{1}\rangle = -\frac{1}{8\pi}\int_{\partial\Omega}\int_{\partial\Omega}|X-Y|\,d\sigma(Y)\,d\sigma(X)<0. \tag{5.104}
$$

Since $\dot{S}\Lambda=0$ we obtain $S\Lambda_0=-c\,\dot{S}\dot{1}$, and thus, using (5.103), the self-adjointness of the operator $\dot{S}$, and (5.104),

$$
0\leq\langle\dot{S}\Lambda_0,\Lambda_0\rangle=-c\langle\dot{S}\dot{1},\Lambda_0\rangle=-c\langle\dot{1},\dot{S}\Lambda_0\rangle=c^2\langle\dot{S}\dot{1},\dot{1}\rangle\leq0. \tag{5.105}
$$

Consequently $c=0$ which forces $\Lambda\in\dot{\mathcal{P}}_0^{\perp}$. With this in hand, the fact that the operator $\dot{S}$ in (5.98) is an isomorphism give $\Lambda=0$. This finishes the proof of the theorem. □

Theorem 5.9 and (5.72) then yield the following.

Theorem 5.10. *Let Ω be an arbitrary bounded Lipschitz domain in $\mathbb{R}^3$. Assume that $1<q<\infty$ and $(s,1/p)\in(0,1)\times(0,1)$ are such that one of the conditions in (1.1) are satisfied. Then the unique solution of (1.2) can be expressed as*

$$u(X) = Bf(X) + \dot{\mathcal{S}}\Big(\dot{S}^{-1}\dot{g}\Big)(X), \qquad X \in \Omega, \tag{5.106}$$

where f is the datum in Ω, B is the volume potential with kernel $B(X-Y)$ (where $B(X)$ is as in (2.143)), and $\dot{g} := \dot{f} - \mathrm{Tr}_{\#}Bf$, where $\dot{f} = (f_0, f_1, f_2, f_3)$ is the Whitney array of boundary data.

A similar integral representation is valid for the solution of (1.5).

Let us point out that, as a corollary of Theorem 5.10 and the Hahn-Banach Theorem, formula (1.6) in the introduction holds.

In closing, we wish to comment on the significance of the invertibility of the operator in (5.98) in the context of the following exterior boundary value problem

$$\begin{cases} \Delta^2 u = 0 \quad \text{in} \quad \Omega_-, \\ \mathrm{Tr}\, u = f_0 \in B^{p,q}_s(\partial\Omega), \\ \mathrm{Tr}\,(\partial_j u) = f_j \in B^{p,q}_s(\partial\Omega), \ 1 \leq j \leq 3, \\ u \in B^{p,q}_{s+\frac{1}{p}+1}(\bar{\Omega}_-; loc), \end{cases} \tag{5.107}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^3$. Compared to (5.43), this time we desire to solve (5.107) in the class of functions u satisfying the decay condition

$$\begin{array}{l} (\partial^\alpha u)(X) = O(|X|^{-|\alpha|}) \quad \text{as} \quad X \to \infty, \\ \text{for every multi-index } \alpha \text{ with } |\alpha| \leq 3. \end{array} \tag{5.108}$$

The claim is that (5.107)-(5.108) is well-posed whenever $\dot{f} := (f_0, f_1, f_2, f_3)$ belongs to $W\!A\big(B^{p,q}_s(\partial\Omega)\big)$ with p, q, s as in (5.97).

The idea is to look for a solution in the form

$$u(X) = C + \dot{\mathcal{S}}\Lambda(X), \qquad X \in \mathbb{R}^3 \setminus \bar{\Omega}, \tag{5.109}$$

for a suitable functional $\Lambda \in W\!A\big(B^{p',q'}_{1-s}(\partial\Omega)\big)^* \cap \dot{\mathcal{P}}_0^{\perp}$ and constant C. In concert with Proposition 5.7, this ensures that the decay condition (5.108) is automatically satisfied (this would have not necessarily been the case had we used Theorem 5.9 and taken $u(X) = \dot{\mathcal{S}}(\dot{S}^{-1}\dot{f})(X))$. Moreover, given that (5.98) is invertible, Λ and C can be chosen so that (5.109) also solves (5.107).

To prove uniqueness, note that if u is a solution of the homogeneous version of (5.107) satisfies and $v := u|_{\Omega_R}$ where $\Omega_R := B_R(0) \setminus \bar{\Omega}$ with $R > 0$ sufficiently large, then $v \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega_R)$ and $\mathrm{Tr}_{\#,R}v$, the trace of v in the sense of (4.76) relative to $\partial\Omega_R$, belongs to $W\!A\big(B^{2,2}_{1/2}(\partial\Omega_R)\big)$. By the well-posedness result for bounded Lipschitz domains, it follows that $v \in B^{2,2}_2(\Omega_R)$, i.e. $u \in B^{2,2}_2(\bar{\Omega}_-; loc)$. Having established this, and granted the decay condition (5.108), the same type of reasoning that led to (5.103) yields $\int_{\mathbb{R}^3\setminus\bar\Omega}|\nabla\nabla u|^2\,dX = \int_{\partial\Omega}\langle \mathrm{Tr}_{\#}u, \partial_\nu^{\#,-}u\rangle\,d\sigma = 0$. Hence, u is locally a linear function in $\mathbb{R}^3\setminus\bar\Omega$, which vanishes in the unbounded component of $\mathbb{R}^3\setminus\bar\Omega$. On the bounded components of $\mathbb{R}^3\setminus\bar\Omega$ we can use the fact that u is harmonic with zero trace in order to conclude that $u\equiv 0$ in $\mathbb{R}^3\setminus\bar\Omega$.

5.3 Sharpness of main results

The following is a consequence of Lemma 1 on p. 44 in [48].

Lemma 5.11. *Assume that $a>0$, $0<p,q\leq\infty$, $s>n(1/p-1)_+$, and fix a function $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\psi\equiv 1$ on $B(0,1)$. The the following equivalences are true:*

$$\psi(X)|X|^a\in B_s^{p,q}(\mathbb{R}^n)\Longleftrightarrow \textit{either } s<\tfrac{n}{p}+a,\ \textit{ or } s=\tfrac{n}{p}+a \textit{ and } q=\infty, \tag{5.110}$$

and

$$\psi(X)|X|^a\in F_s^{p,q}(\mathbb{R}^n)\Longleftrightarrow s<\tfrac{n}{p}+a. \tag{5.111}$$

We now discuss the analogue of the off-diagonal estimates for the Green operator associated with the Dirichlet Laplacian in Lipschitz domains, established by B.E.J. Dahlberg in [10]. Concretely, we have:

Theorem 5.12. *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain which is star-like with respect to the origin. Then there exists $\varepsilon=\varepsilon(\Omega)>0$ with the property that if*

$$\tfrac{3}{3+\varepsilon}<p<\tfrac{3}{2-\varepsilon}\quad \textit{and}\quad \tfrac{1}{q}=\tfrac{1}{p}-\tfrac{1}{3} \tag{5.112}$$

then the operator

$$\mathbf{G}: F_{-1}^{p,2}(\Omega)\longrightarrow F_{2,z}^{q,2}(\Omega) \tag{5.113}$$

is well-defined and bounded. Furthermore, as far as the well-definiteness of (5.113) is concerned, the conditions in (5.112) are optimal.

Proof. Consider an arbitrary function $f\in F_{-1}^{p,2}(\Omega)$ and construct $w\in F_3^{p,2}(\Omega)$ such that $\Delta^2 w=f$ and $\|w\|_{F_3^{p,2}(\Omega)}\leq C\|f\|_{F_{-1}^{p,2}(\Omega)}$. Then $\mathbf{G}f=w-u$, where u solves $\Delta^2 u=0$ in Ω, $\mathrm{Tr}_{\#}u=\mathrm{Tr}_{\#}w$ on $\partial\Omega$. Note that $w\in F_3^{p,2}(\Omega)\hookrightarrow F_2^{q,2}(\Omega)$ if $1/q=1/p-1/3$ and that, accordingly, $\mathrm{Tr}_{\#}w\in W\!A\big(B_{1-1/q}^{q,q}(\partial\Omega)\big)$. Then Theorem 1.1 implies that $u\in F_2^{q,2}(\Omega)$ as well, granted that the point with coordinates $(1-1/q,q)$ belongs to the pentagonal region described in (5.14). A simple analysis shows that this is always the case whenever $\frac{3}{2+\varepsilon}<q<\frac{3}{1-\varepsilon}$, for some $\varepsilon=\varepsilon(\Omega)>0$. The bottom line is that

$$f \in F^{p,2}_{-1}(\Omega) \Longrightarrow \mathbf{G}f \in F^{q,2}_{2,z}(\Omega) \quad \text{if} \quad \tfrac{3}{2+\varepsilon} < q < \tfrac{3}{1-\varepsilon}, \ \ \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{3}. \tag{5.114}$$

Next, (1.13) with $\alpha = 0$, $p = q = 2$, and classical embeddings give

$$\mathbf{G} : F^{p,2}_{-1}(\Omega) \longrightarrow F^{p^*,2}_{2,z}(\Omega) \quad \text{if} \quad \tfrac{3}{3+\varepsilon} < p < 1, \ \ \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{3}. \tag{5.115}$$

Interpolating by the complex method between (5.114) and (5.115) then yields (5.113) in full, as long as $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$ and $1 < q < \frac{3}{1-\varepsilon}$. Note that the last condition precisely amounts to asking that (5.112) holds.

We shall now show that the conditions in (5.112) are optimal. To this end, we record the following consequence of the construction in Lemma 10.6 of [43]. For each $\theta \in (0, \pi)$ there exist a bounded, star-like Lipschitz domain Ω_θ in $\mathbb{R}^3$ such that

$$\Omega_\theta \cap B(0,1) = \Big\{X = (x_1, x_2, x_3) \in B(0,1) :\ x_3 < (\cot\theta)\sqrt{x_1^2 + x_2^2}\Big\} \tag{5.116}$$

and for which there exists a function $u : \Omega_\theta \to \mathbb{R}$ satisfying

$$u \in C^\infty \text{ in } \overline{\Omega_\theta} \text{ away from the origin,} \tag{5.117}$$

$$u(X) \equiv |X|^{\lambda(\theta)}\varphi(X/|X|) \text{ for } X \text{ near } 0, \tag{5.118}$$

$$\varphi \in C^\infty(S^2) \text{ and } \lambda(\theta) \searrow 1 \ \text{ as } \ \theta \searrow 0, \tag{5.119}$$

$$\Delta^2 u \in C^\infty(\overline{\Omega_\theta}), \quad u = \partial_\nu u = 0 \text{ on } \partial\Omega_\theta. \tag{5.120}$$

Note, on the one hand, that conditions (5.117)-(5.120) and Lemma 5.11 ensure that $u \in W^{2,2}(\Omega)$ so if we set $f := \Delta^2 u \in C^\infty(\overline{\Omega_\theta})$, then $Gf = u$. On the other hand, (5.117)-(5.120) and Lemma 5.11 also give

$$u \in F^{q,2}_{2}(\Omega) \Longleftrightarrow 2 < \tfrac{3}{q} + \lambda(\theta) \Longleftrightarrow p < \tfrac{3}{3-\lambda(\theta)} \searrow 3 \text{ as } \theta \searrow 0, \tag{5.121}$$

if $1/q = 1/p - 1/3$. This proves that the upper bound for p in (5.112) is sharp.

To show that the lower bound is also optimal, we shall rely on duality; cf. (4.35)-(4.36). Also, with $C^{s+1}_z(\bar{\Omega}) := \{u \in C^{s+1}(\bar{\Omega}) :\ u|_{\partial\Omega} = (\nabla u)|_{\partial\Omega} = 0\}$, for $s \in (0,1)$, we have (cf. [4], [30])

$$\Big(F^{p,2}_{-1}(\Omega)\Big)^* = C^{s+1}_z(\bar{\Omega}) \quad \text{if} \ \ \tfrac{n}{n+1} < p < 1 \ \ \text{and} \ \ s = n\Big(\tfrac{1}{p} - 1\Big). \tag{5.122}$$

Dualizing the result about the operator (5.113) then yields the following consequences (of intrinsic interest):

$$\mathbf{G} : F^{p,2}_{-2}(\Omega) \longrightarrow F^{q,2}_{1,z}(\Omega) \quad \text{if } \tfrac{3}{2+\varepsilon} < p < 3 \quad \text{and} \quad \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{3}, \tag{5.123}$$

and

$$\mathbf{G} : F^{p,2}_{-2}(\Omega) \longrightarrow C^{\alpha+1}_{z}(\bar{\Omega}) \quad \text{if } 3 < p < \tfrac{3}{1-\varepsilon} \quad \text{and} \quad \alpha = 1 - \tfrac{3}{p}. \tag{5.124}$$

However, (5.117)-(5.120) and Lemma 5.11 give that

$$u \in C^{\alpha+1}(\bar{\Omega}) \Longleftrightarrow \alpha + 1 < \tfrac{3}{p} + \lambda(\theta) \Longleftrightarrow p < \tfrac{6}{2-\lambda(\theta)} \searrow 3 \text{ as } \theta \searrow 0, \tag{5.125}$$

if $\alpha = 1 - 3/p$, which proves that the upper bound for p in (5.124) is sharp. This, in turn, can be traced back to the lower bound for p in (5.112) being sharp. □

The fact that the range of indices in Theorem 1.1 dictates the range of indices in Theorem 5.12 which, in turn, is sharp, shows that Theorem 1.1 is in the nature of best possible. A more direct argument is as follows. The counterexample in (5.117)-(5.120) readily gives that the region in Figure 1 described by $\frac{1}{p} < \frac{\alpha+2}{3}$ cannot contain points $(\alpha, 1/p)$ which are "good" (in the sense of (1.12)-(1.13)) for *all* bounded Lipschitz domains in $\mathbb{R}^3$. Since, by duality, stability and interpolation,

$$\begin{aligned} &\text{the subregion of } \{(\alpha, 1/p) : \min\{1, \alpha+2\} > \tfrac{1}{p} > \max\{0, \alpha+1\}\} \\ &\text{for which (1.12)-(1.13) are bounded operators is open, convex} \\ &\text{as well as symmetric with respect to the point } (-1, \tfrac{1}{2}), \end{aligned} \tag{5.126}$$

it follows that the bound $\frac{1}{p} > \frac{\alpha}{3} + 1$ is also, generally speaking, sharp.

We conclude with a remark on the nature of the Green function for the bi-Laplacian in Lipschitz domains. Let $G(X,Y)$ be the integral kernel of the Green operator $\mathbf{G}$, i.e., the Green function associated with the bi-Laplacian in Ω. That is,

$$\mathbf{G}f(X) = \int_{\Omega} G(X,Y) f(Y)\, dY, \qquad X \in \Omega, \tag{5.127}$$

where

$$\begin{aligned} &\Delta^2_Y[G(X,Y)] = \delta_X(Y) \quad \text{for } X, Y \in \Omega, \quad \text{and} \\ &G(X,Y) = \partial_{\nu(Y)}[G(X,Y)] = 0 \quad \text{for } X \in \Omega,\ Y \in \partial\Omega. \end{aligned} \tag{5.128}$$

Denote by $\nabla^j_X \nabla^k_Y \mathbf{G}$ the integral operator with kernel $\nabla^j_X \nabla^k_Y [G(X,Y)]$. Theorem 5.12 then yields the following. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the following are bounded mappings:

$$\nabla_X^2\nabla_Y^1\mathbf{G} : h^p(\Omega) \longrightarrow L^q(\Omega), \quad \text{if} \quad \tfrac{3}{3+\varepsilon} < p < \tfrac{3}{2-\varepsilon} \quad \text{and} \quad \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{3}, \tag{5.129}$$

$$\nabla_X^2\nabla_Y^1\mathbf{G} : L^1(\Omega) \longrightarrow L^{\frac{3}{2},\infty}(\Omega), \qquad \text{the Lorentz space.} \tag{5.130}$$

Above, $h^p(\Omega) := F_0^{p,2}(\Omega)$ is the local Hardy space in Ω when $p \leq 1$, and the Lebesgue space $L^p(\Omega)$ when $p > 1$. Indeed, the fact that the operator in (5.129) is bounded follows from Theorem 5.12 and embeddings. In turn, this and real interpolation gives that $\nabla_X^2\nabla_Y^1\mathbf{G}$ maps the weak Hardy space $h^{1,\infty}(\Omega)$ boundedly into the Lorentz space $L^{\frac{3}{2},\infty}(\Omega)$. Since $L^1(\Omega) \hookrightarrow h^{1,\infty}(\Omega)$, we may conclude that the operator in (5.130) is also bounded.

Thus, heuristically, $\nabla_X^2\nabla_Y^1\mathbf{G}$ behaves like a fractional integral operator of order 1 in $\mathbb{R}^3$, albeit only for a restricted range of indices. In particular, the estimate

$$|\nabla_X^2\nabla_Y[G(X,Y)]| \leq C|X-Y|^{-2}, \qquad X, Y \in \Omega, \tag{5.131}$$

may fail in arbitrary Lipschitz domains.

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